HOBET

I. Harmonic oscillator-based effective theory (HOBET)

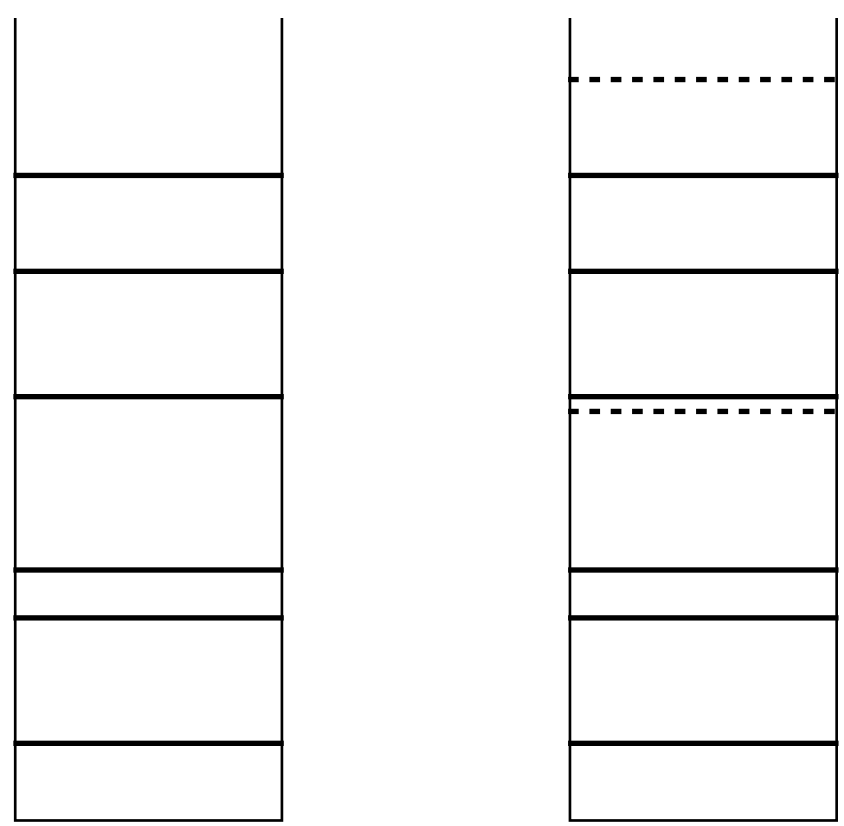
II. Review of its implementation

III. Two new questions addressed

Effective theory

- For decades we have thought about effective nuclear theory: the division of the Hilbert space into an included space P and an excluded space Q, and the determination of a $H^{\rm eff}$ in P (including models liked the SM)

- The first catastrophic failure of this theory was recognized in the early 1970s: perturbative efforts to generate $H^{\rm eff}$ derailed by intruder states



$4\hbar\omega$ intruder state

$$\langle 4\hbar\omega | H | 2\hbar\omega \rangle >> \Delta E$$

e.g., $P = 0 + 2\hbar\omega$

Nonperturbative corrections to $H^{\rm eff}$

Shucan and Weidenmuller
Barrett and Kirson

There is now a major program based on a nonperturbative approach: ET

- Lots of wonderful things could be said about this approach … so it is admittedly unfair to focus on the negatives … but …

$E = \infty$ UV: QCD

$E = 1\ \mathrm{GeV}$ NN potential

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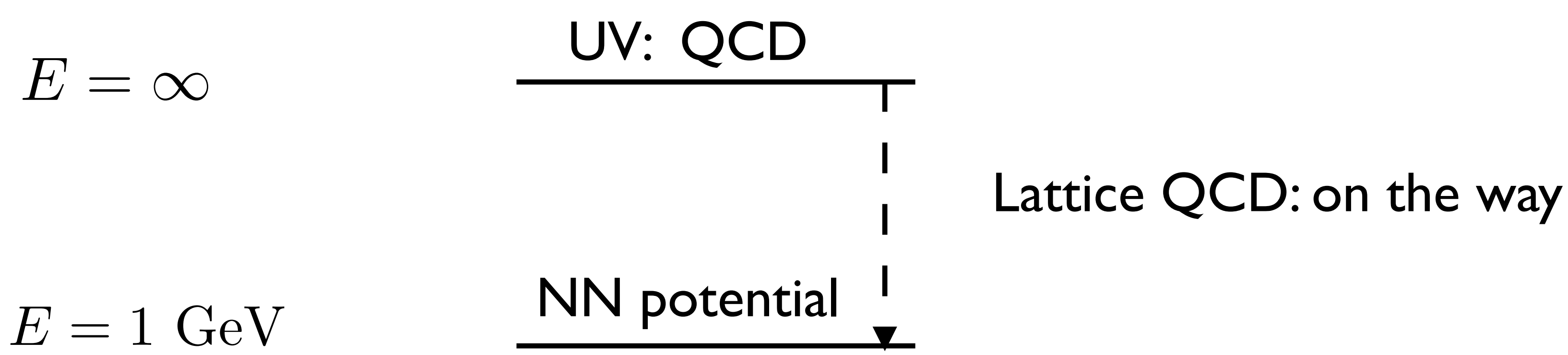

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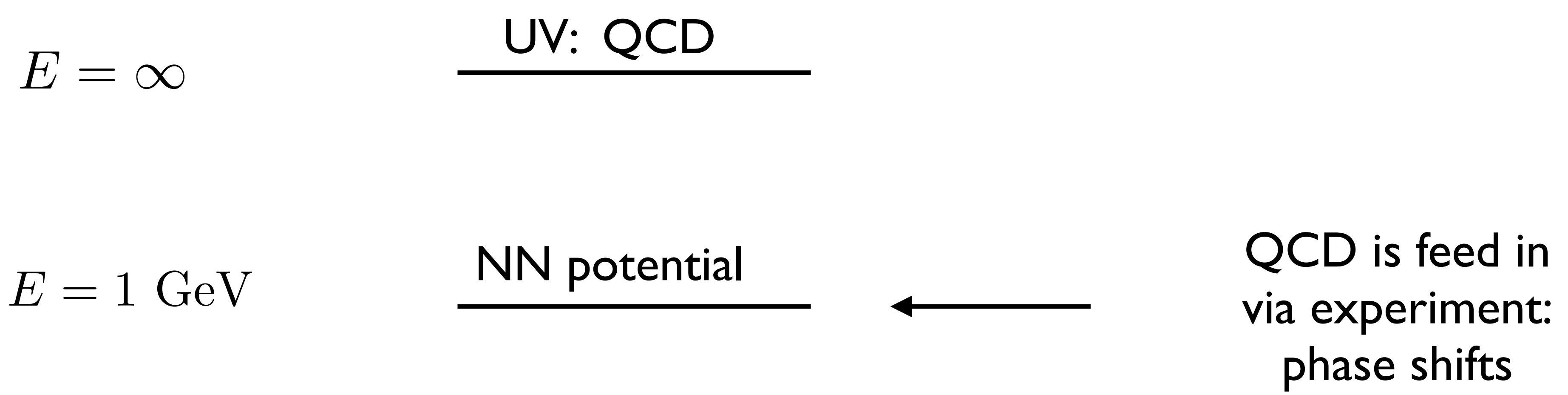

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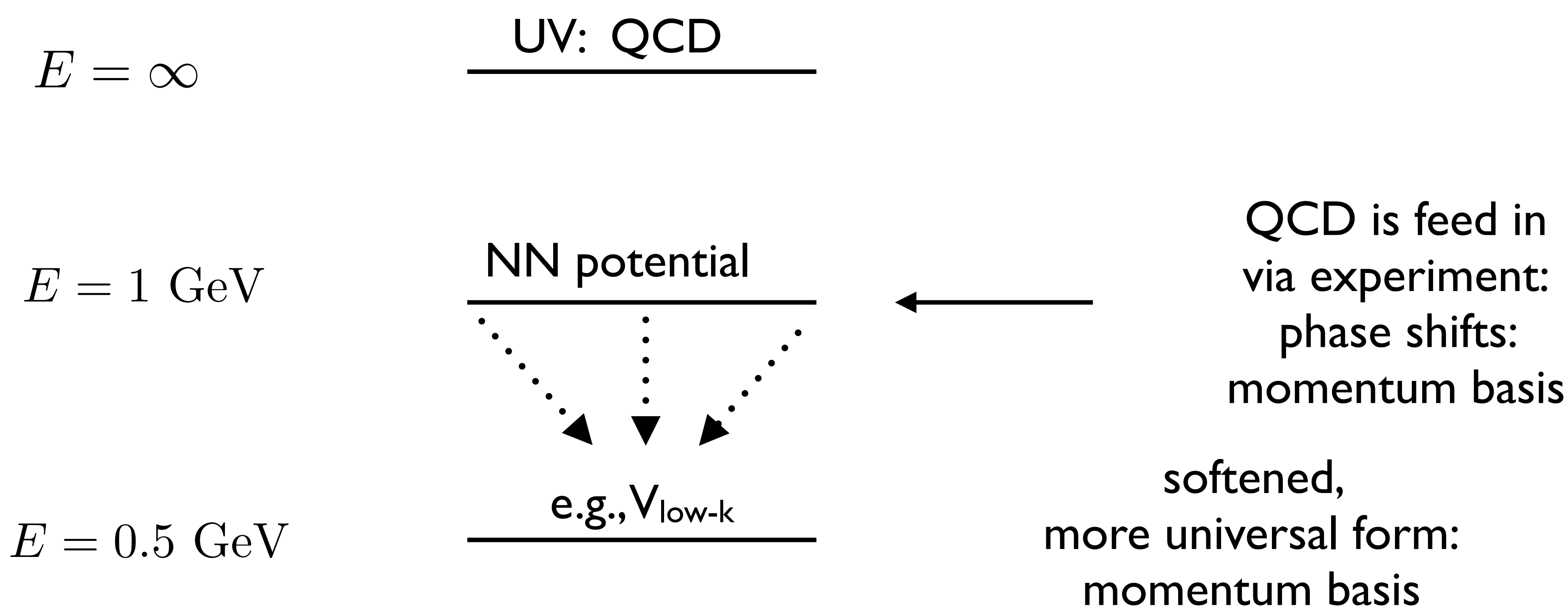

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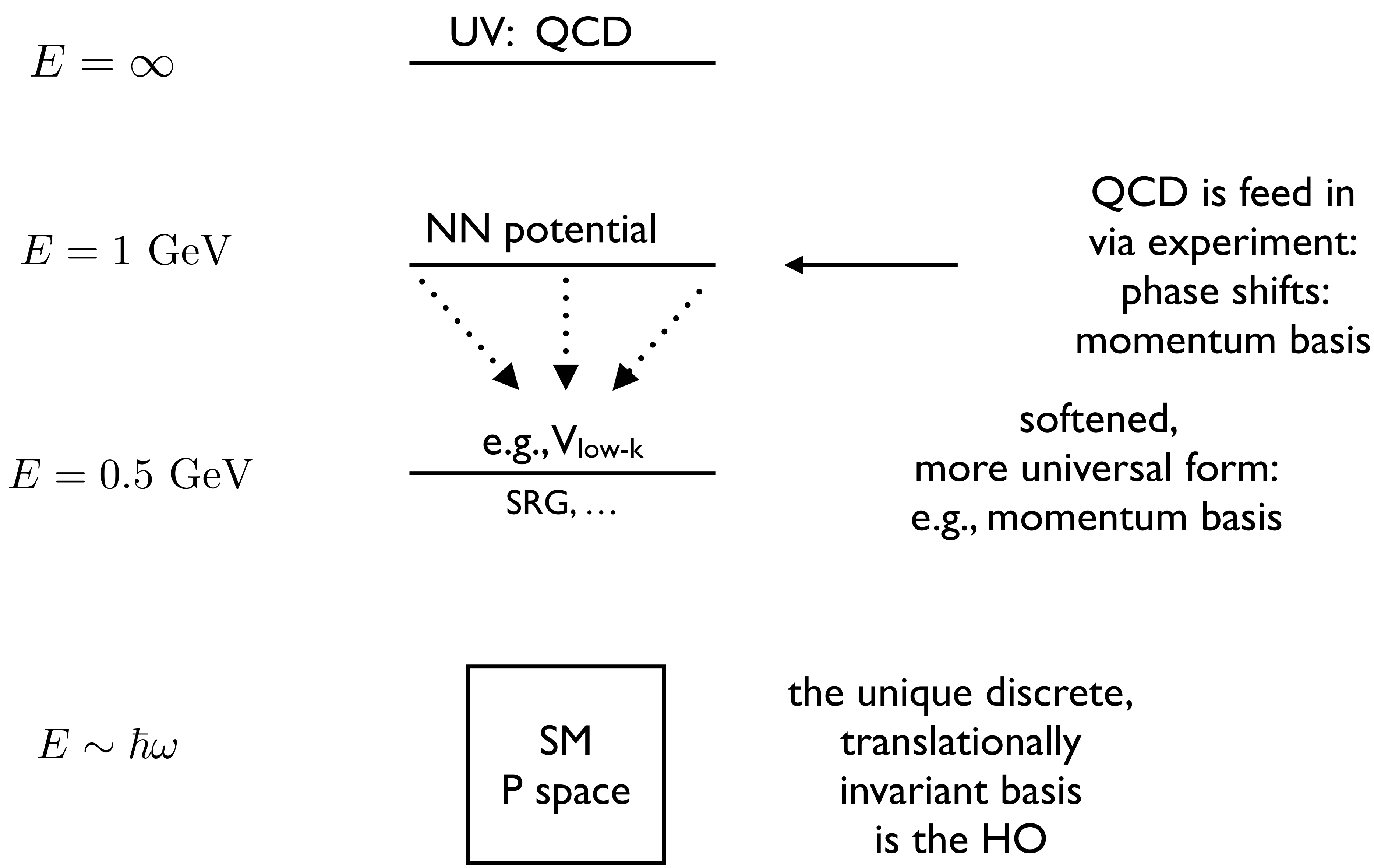

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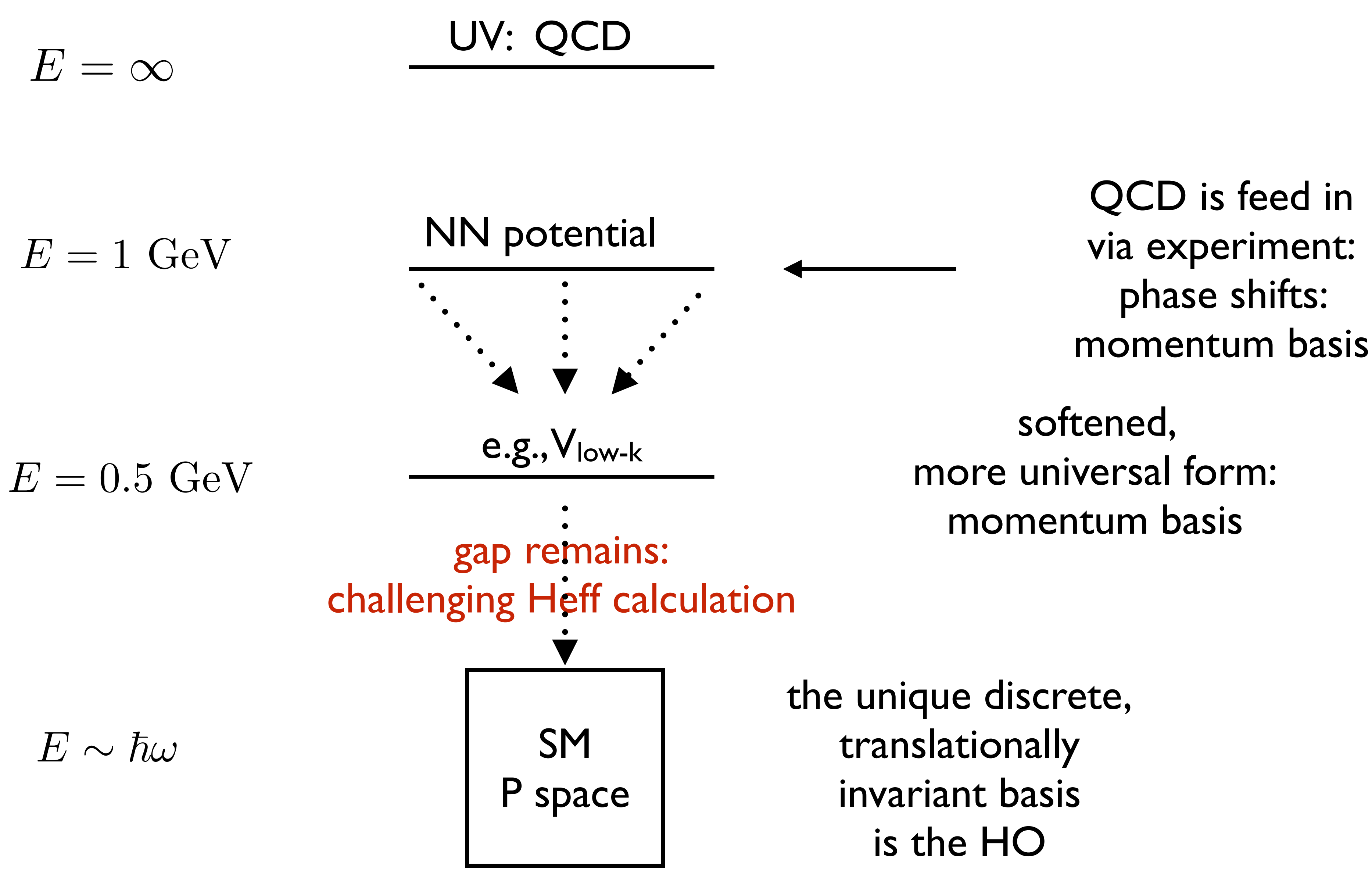
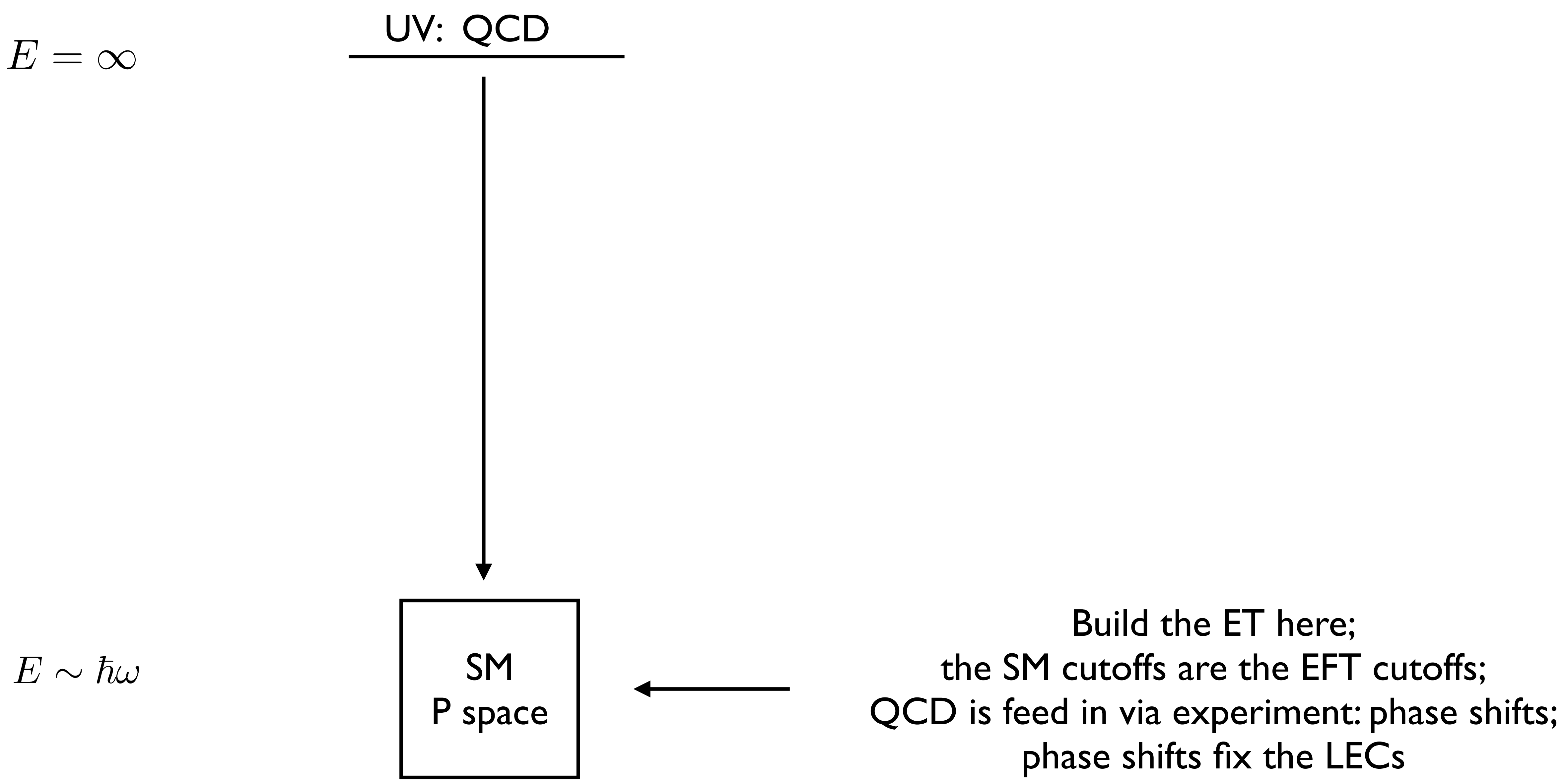
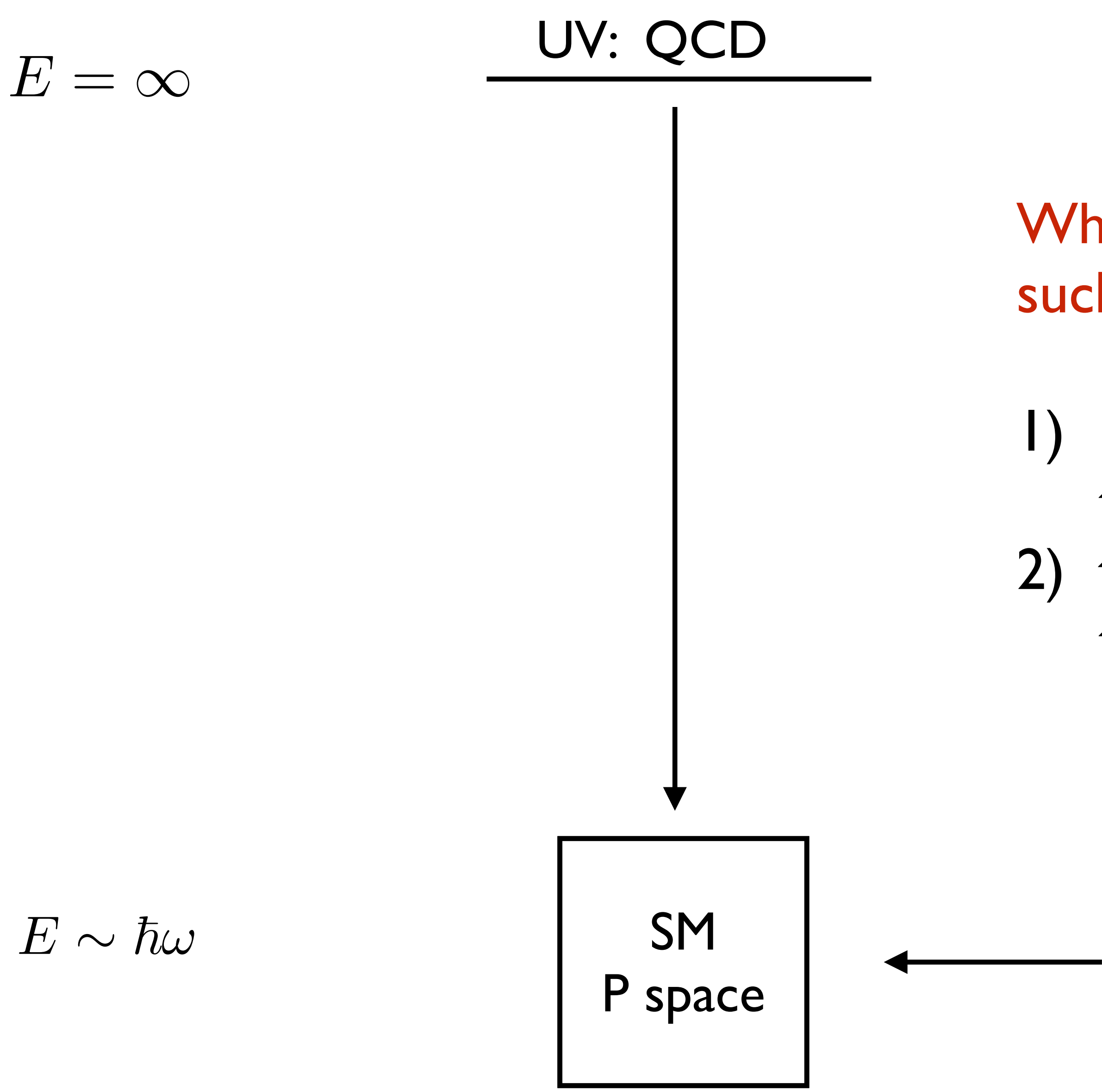


HOBET is intended as a direct reduction from QCD

1) Simplify to a true ET



HOBET was design to simplify such procedures



What is the best starting point for building such a theory?

1) requires a theory that applies equally to bound states and continuum states
2) to use phase shifts, must have a theory that relates H^{eff} to $\delta(E)$

Build the ET here;
the SM cutoffs are the EFT cutoffs;
QCD is feed in via experiment: phase shifts;
phase shifts fix the LECs

Key ideas of HOBET

- Utilize the unique discrete basis that allows one to preserve translational invariance - critical to any true EFT

- Use the energy-dependent Bloch-Horowitz equation
  - ❖ yields exact eigenvalues, exact projections of true wave functions
  - ❖ no intruder-state problem: an infinite number of solution from a finite P
  - ❖ analytically continuous in E: applies equally to bound and continuum states
  - ❖ allows one to make precise connections between LECs and phase shifts, many of which evolve rapidly with E
  - ❖ the "have your cake and eat it to" theorem: the BH equation can be reorganized so the the LECs are energy independent
  - ❖ exact cutoff independence: no dependence on the choice of b, Λ

- The theory is systematic and rapidly convergent at nuclear momentum scales

- Builds in chiral symmetry in a much more elegant way: avoids the tedious short-range pionic expansions of standard chiral EFT

BH Formulation

- Nonrelativistic effective theory that is formulated in a HO P-space: discrete but translationally invariant

- Analytically continuous in E: applies equally to bound states or reactions

- Based on a reorganization of the Bloch-Horowitz equation (WH + Tom Luu). Here $E, \ |\Psi\rangle$ are the full solution,

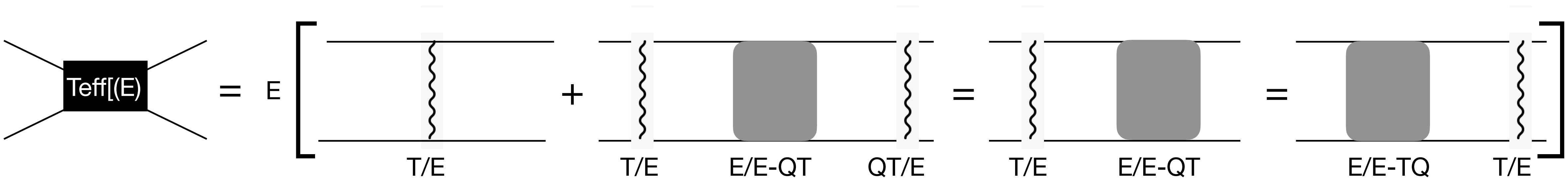
$$H = T + V$$

$$PH^{\mathrm{eff}} P|\Psi\rangle = EP|\Psi\rangle$$

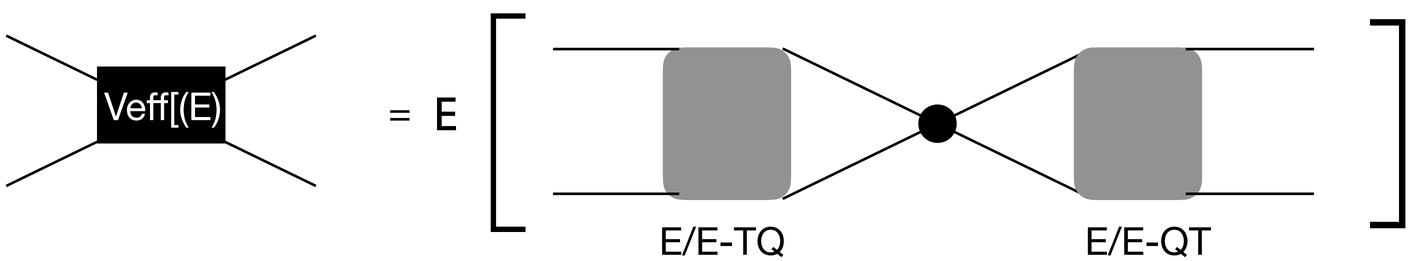
$$H^{\mathrm{eff}} = H + H \frac{1}{E - QH} QH \equiv T^{\mathrm{eff}} + V^{\mathrm{eff}}$$

the reorganization:

Any HO-based EFT must have this form of the
effective kinetic energy operator

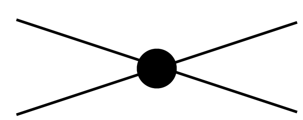


Very energy dependent - but can be rewritten in terms of the free Green’s function E/E-T, known analytically, at the cost of a matrix inversion in P



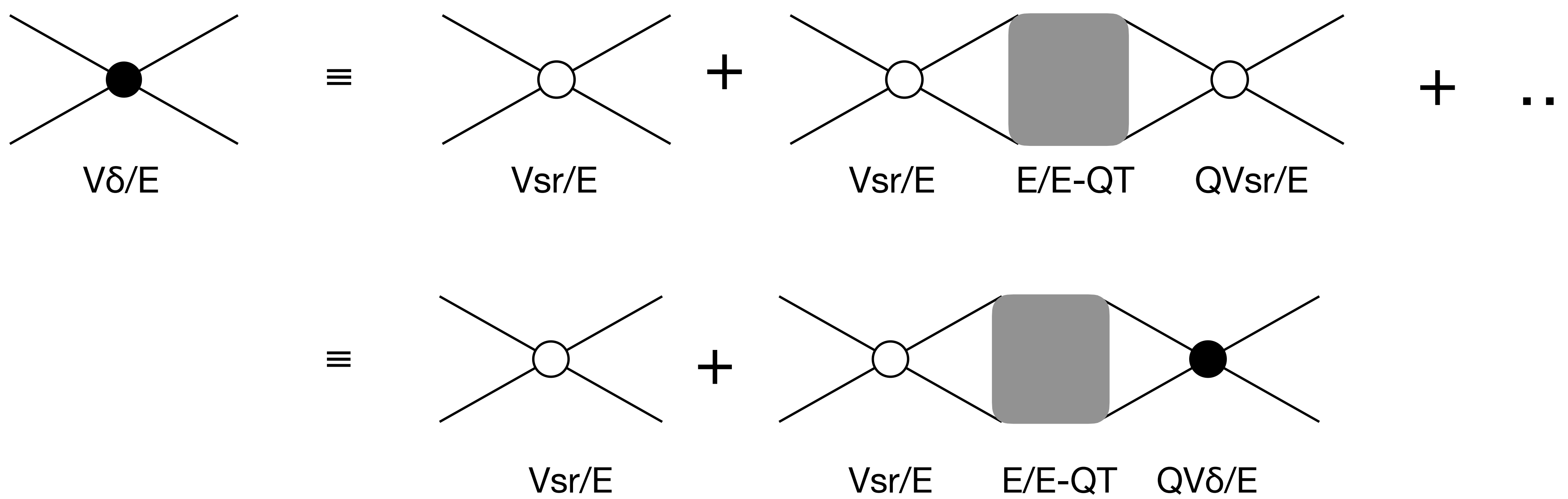
Very energy dependent

But not this - only weakly E-dependent



95% of the energy dependence removed

This is the quantity that can be expanded in short range operators



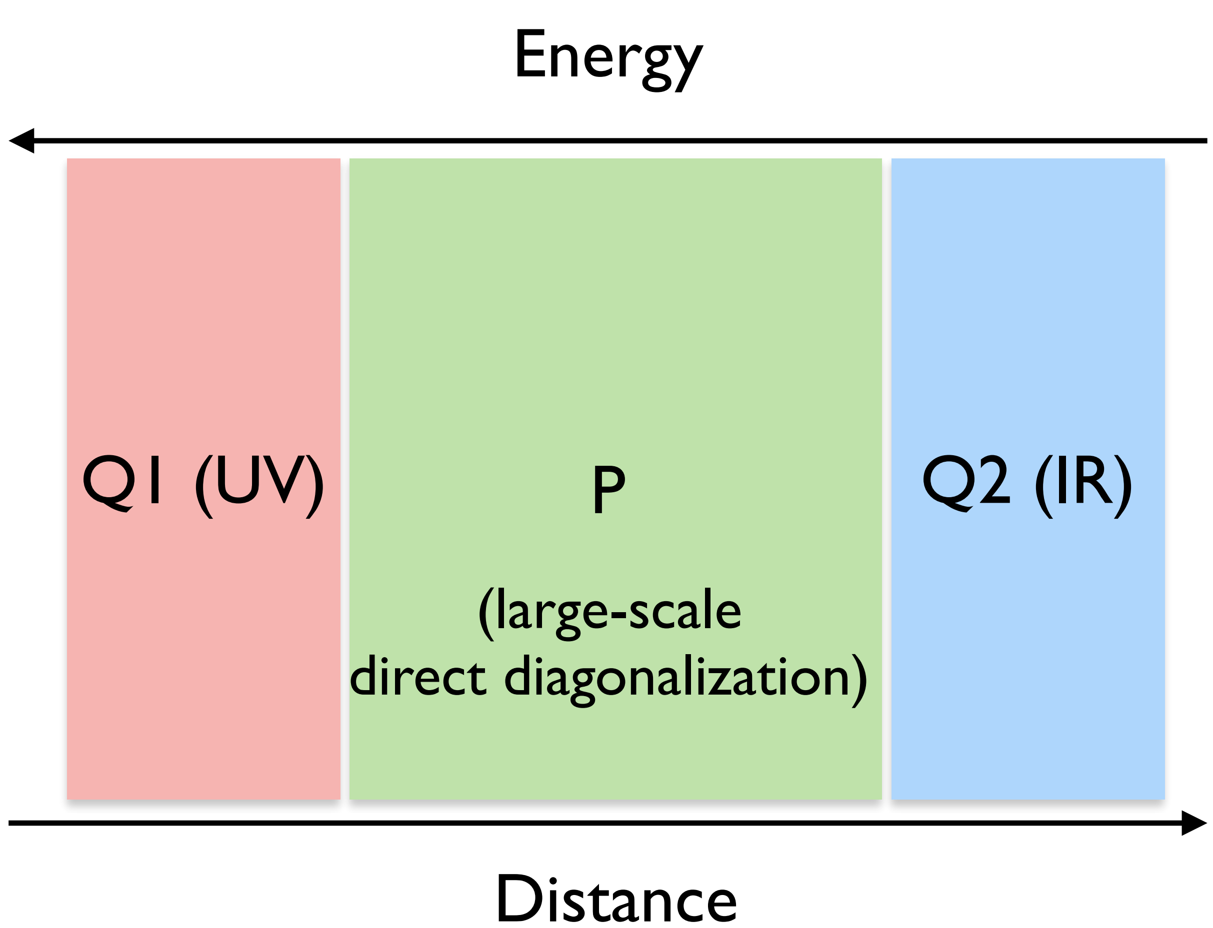
Were we were working in a potential theory, Vsr would be the short-range nuclear potential - which we can expand in HOBET's pion-less and pion-full operators

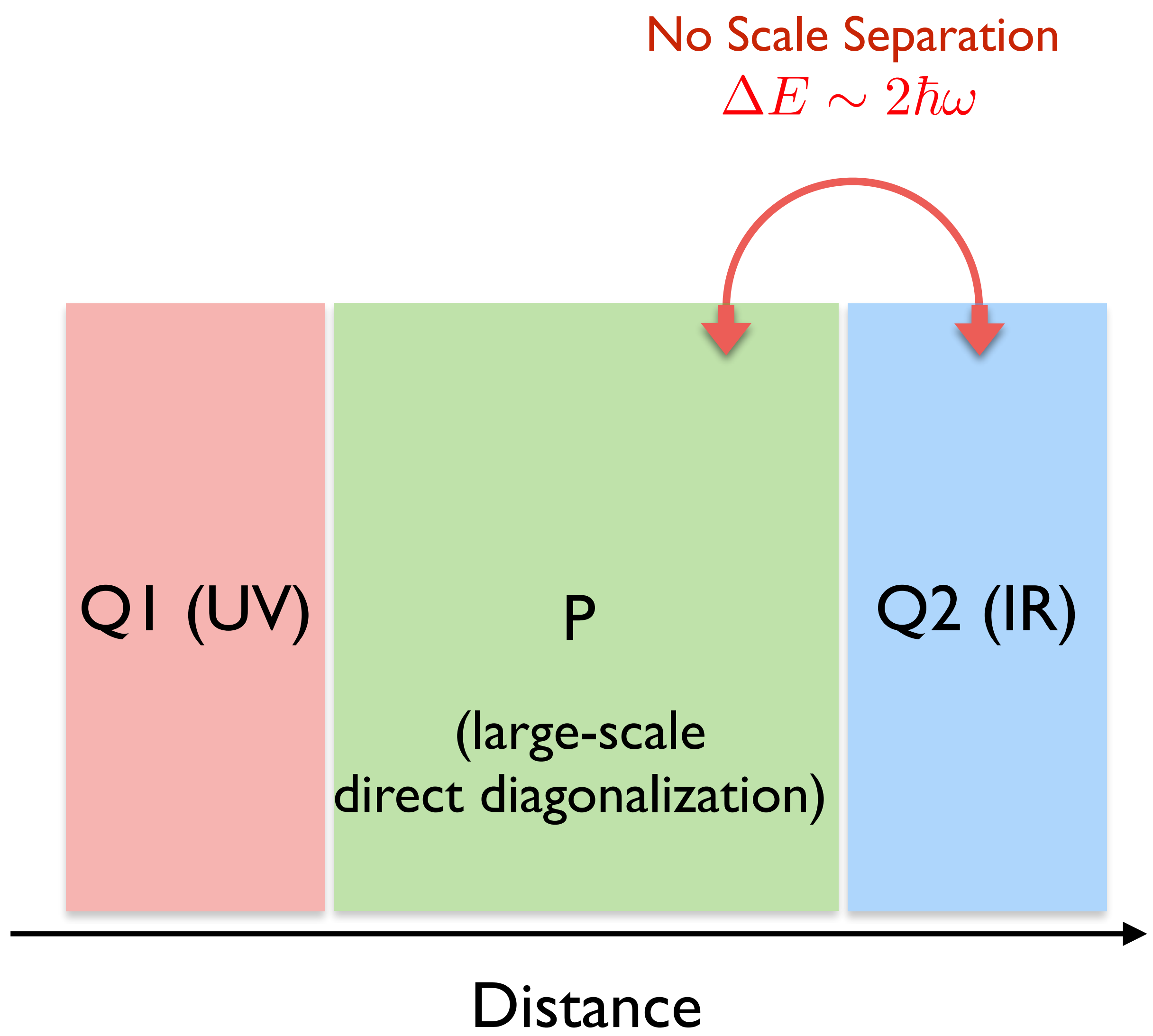
Vsr is slightly energy dependent - 5% of the original energy dependence remains after this reoganization: Easily absorbed into the momentum-dependence of HOBET's contact-gradient operator expansion

Consequently $H^{\mathrm{eff}}(E)$ can be expressed in terms of a single set of energy-independent LECs

UV-IR Separation and Energy-Dependence

- Nuclear ground states are a compromise between the UV and the IR: kinetic energy is minimized by delocalization; potential energy is minimized by localizing at scales $\sim 1/m_\pi$
- Corrections due to omitted IR and UV physics are roughly comparable in importance — but differ greatly in their consequences for ET





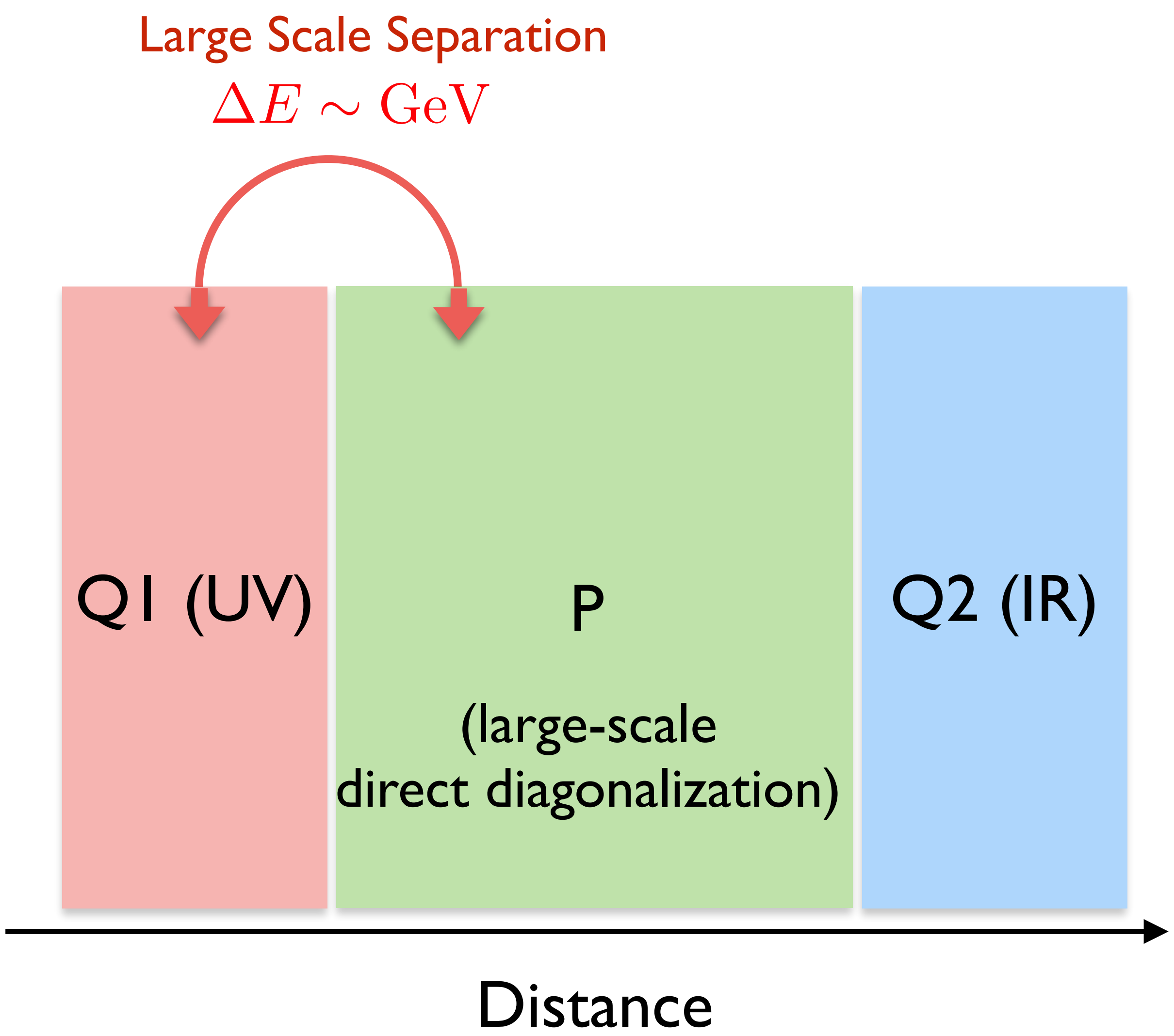
Coupling between P and Q2 is via the K.E. operator

$\vec{\nabla}^2$ connects neighboring shells

this means small energy denominators, highly energy dependent corrections

must be treated - but can be quasi-analytically

IR propagation enhanced because nuclei barely bound



Coupling between P and Q1 is via short-range strong interactions

Large energy denominators: energy independent corrections

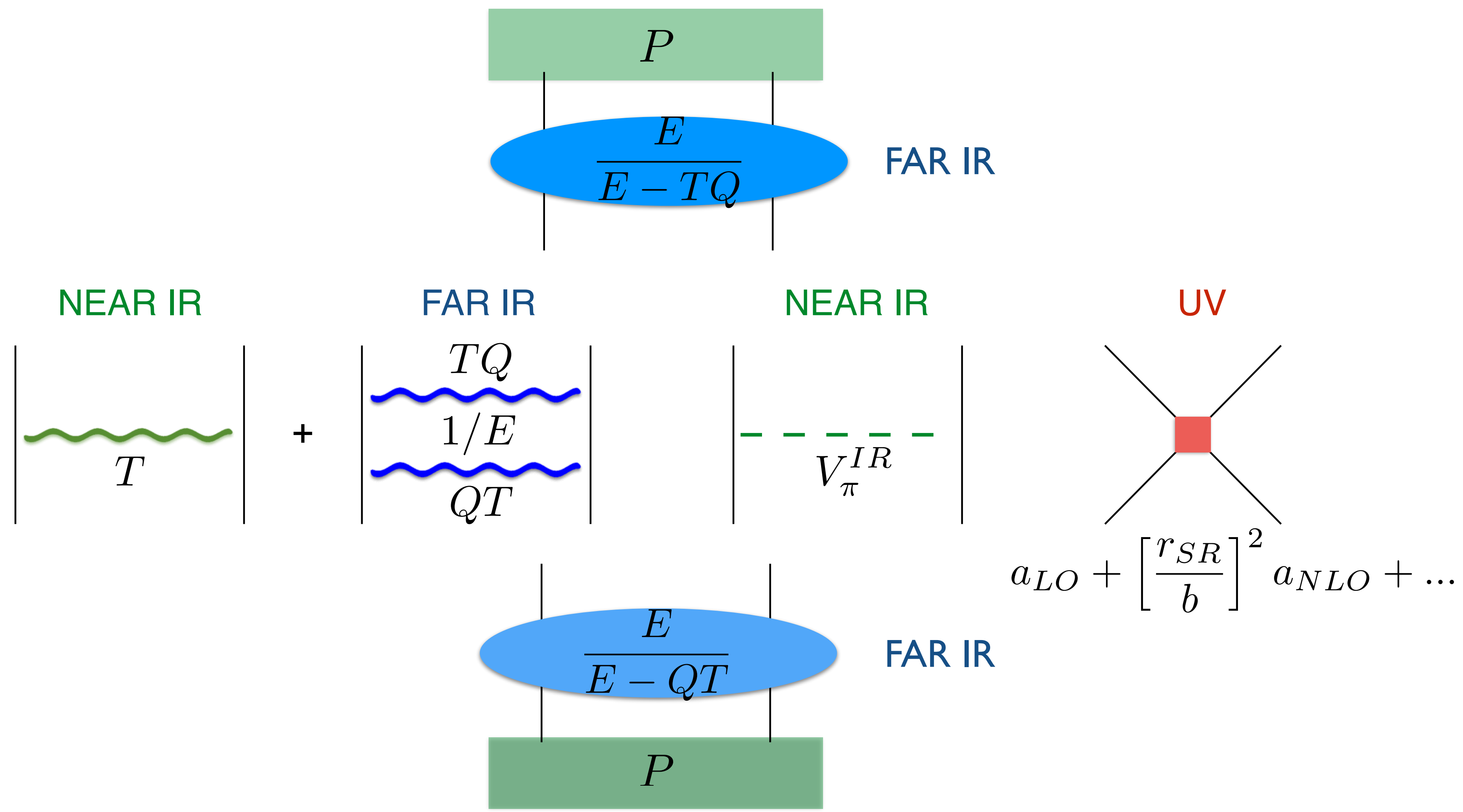
Can be treated by a standard short range expansion

- Build the effective theory:

$$H^{\text{eff}} = G_{TQ}\left[T + T\frac{Q}{E}T + V + VG_{QH}QV\right]G_{QT}$$

$$V + V\ G_{QH}\ QV \rightarrow \begin{cases} V_\delta & \text{pionless} \\ V_\pi^{\text{IR}} + V_\delta & \text{pionful} \end{cases}$$

no reference to SR potential remains



Fixing the LECs: HOBET's short-range expansion is one in HO quanta:

$$(a_x^\dagger, a_y^\dagger, a_z^\dagger): \qquad a_i \equiv \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial r_i} + r_i\right) \qquad a_i \equiv \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial r_i} + r_i\right)$$

$$\boldsymbol{r} = \frac{1}{\sqrt{2}b}(\boldsymbol{r}_1 - \boldsymbol{r}_2) \qquad a_M^\dagger = \hat{e}_M \cdot \boldsymbol{a}^\dagger \qquad \tilde{a}_M = (-1)^M a_{-M}$$

- From these operators one can construct nodal and angular momentum raising and lowering operators

$$\tilde{\mathbf{a}} \odot \tilde{\mathbf{a}}\, |n\ell m\rangle = -2\sqrt{(n-1)\,(n+\ell-1/2)}\; |n-1\,\ell m\rangle$$

$$[[\tilde{\mathbf{a}} \otimes \tilde{\mathbf{a}} \otimes \cdots \otimes \tilde{\mathbf{a}}]_\ell \otimes |n\ell\rangle]_{00} = (-1)^\ell\, 2^{\ell/2} \sqrt{\frac{l!}{(2\ell-1)!!}\,\frac{\Gamma[n+\ell+\frac{1}{2}]}{\Gamma[n+\frac{1}{2}]}}\; |n00\rangle$$

- The expansion is effectively one around $r \sim b$

- Expansion order is defined in terms of oscillator quanta

$$V_\delta^S = a^{\mathrm{S}}_{LO}\delta(\boldsymbol{r}) + a^{\mathrm{S}}_{NLO}\left(\boldsymbol{a}^\dagger \odot \boldsymbol{a}^\dagger\ \delta(\boldsymbol{r}) + \delta(\boldsymbol{r})\ \tilde{\boldsymbol{a}} \odot \tilde{\boldsymbol{a}}\right) + \cdots$$

$$\delta(\boldsymbol{r}) \equiv \sum_{n'n} d^{00}_{n'n} |n'00\rangle\langle n00| \qquad\qquad d^{00}_{n'n} \equiv \frac{2}{\pi^2}\left[\frac{\Gamma(n'+\frac{1}{2})\Gamma(n+\frac{1}{2})}{(n'-1)!(n-1)!}\right]^{1/2}$$

$$\langle n'(\ell'=0S)JM;TM_T|V_\delta^{\mathrm{S}}|n(\ell=0S)JM;TM_T\rangle = d^{00}_{n'n}\left[a_{LO} - 2[(n'-1)+(n-1)]a^{\mathrm{S}}_{NLO} + \cdots\right]$$

- If we had computed the LECs from a potential, we would have found that the LECs are a non-local generalization of the familiar Talmi integrals

$$\int d\boldsymbol{r}' d\boldsymbol{r}\ r^{2p'} e^{-r'^2/2} Y_{00}(\Omega')\, V(\boldsymbol{r}',\boldsymbol{r})\, r^{2p} e^{-r^2/2} Y_{00}(\Omega)$$

$$a_{LO} \leftrightarrow (p',p) = (0,0) \qquad a_{NLO} \leftrightarrow (p',p) = (0,1) \text{ or } (1,0) \qquad \text{etc.}$$

HOBET's V^{IR}_{π}

- V_π is already regulated by b, Λ
- our LO expansion systematically subtracts out the shortest range Talmi integrals: there is a 1-to-1 correspondence between LECs and Talmi integrals
- it makes no sense to include V_π in any Talmi integral that has an fitted LEC

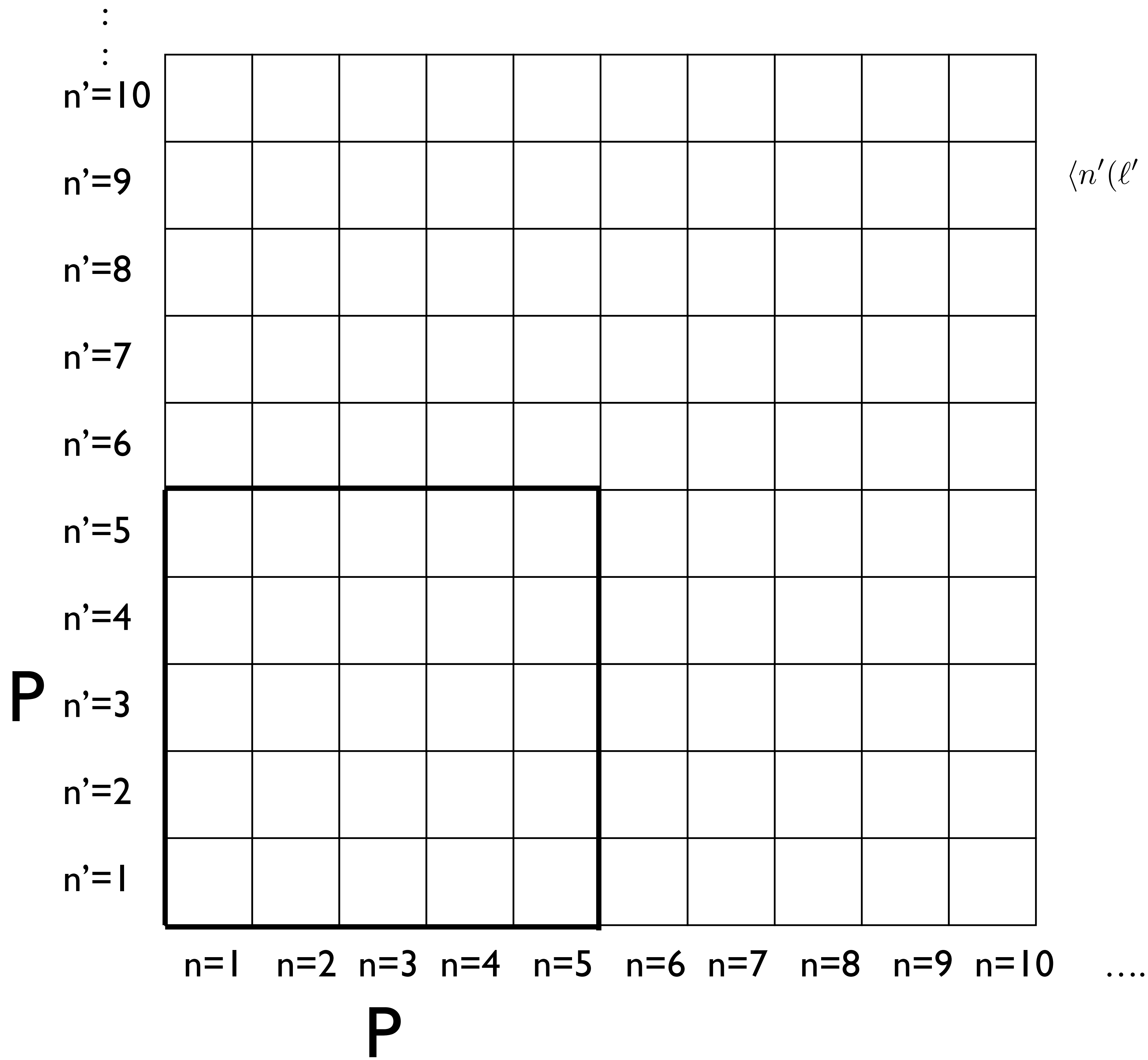
$$V_\pi = V_\pi^{IR} + V_\pi^{UV} \qquad\qquad V_\pi^{UV} = V_\delta(a_{LECs} \to a^\pi_{LECs})$$

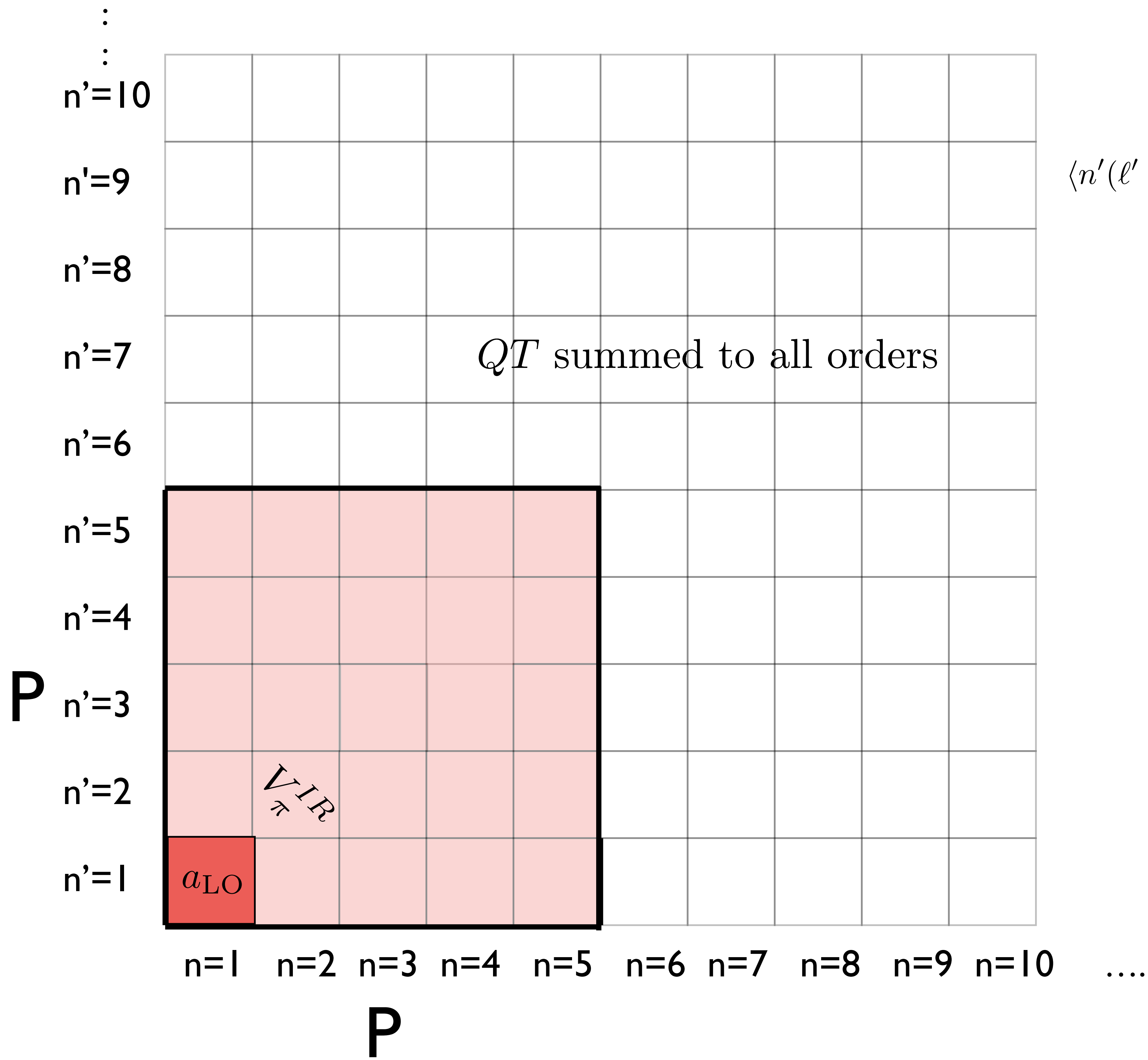
thus the only effect of V_π^{IR} is to correct the long-range Talmi integrals for which there is no LEC

- thus in HOBET the pion is a near-infrared contribution, weak and perturbative: its peak contribution (b=1.7 f) is at 4.1 f

$$\Lambda = 8\hbar\omega$$

$$\langle n'(\ell' = 0S)J;T|\ H^{\mathrm{eff}}\ |n(\ell = 0S)J;T\rangle$$



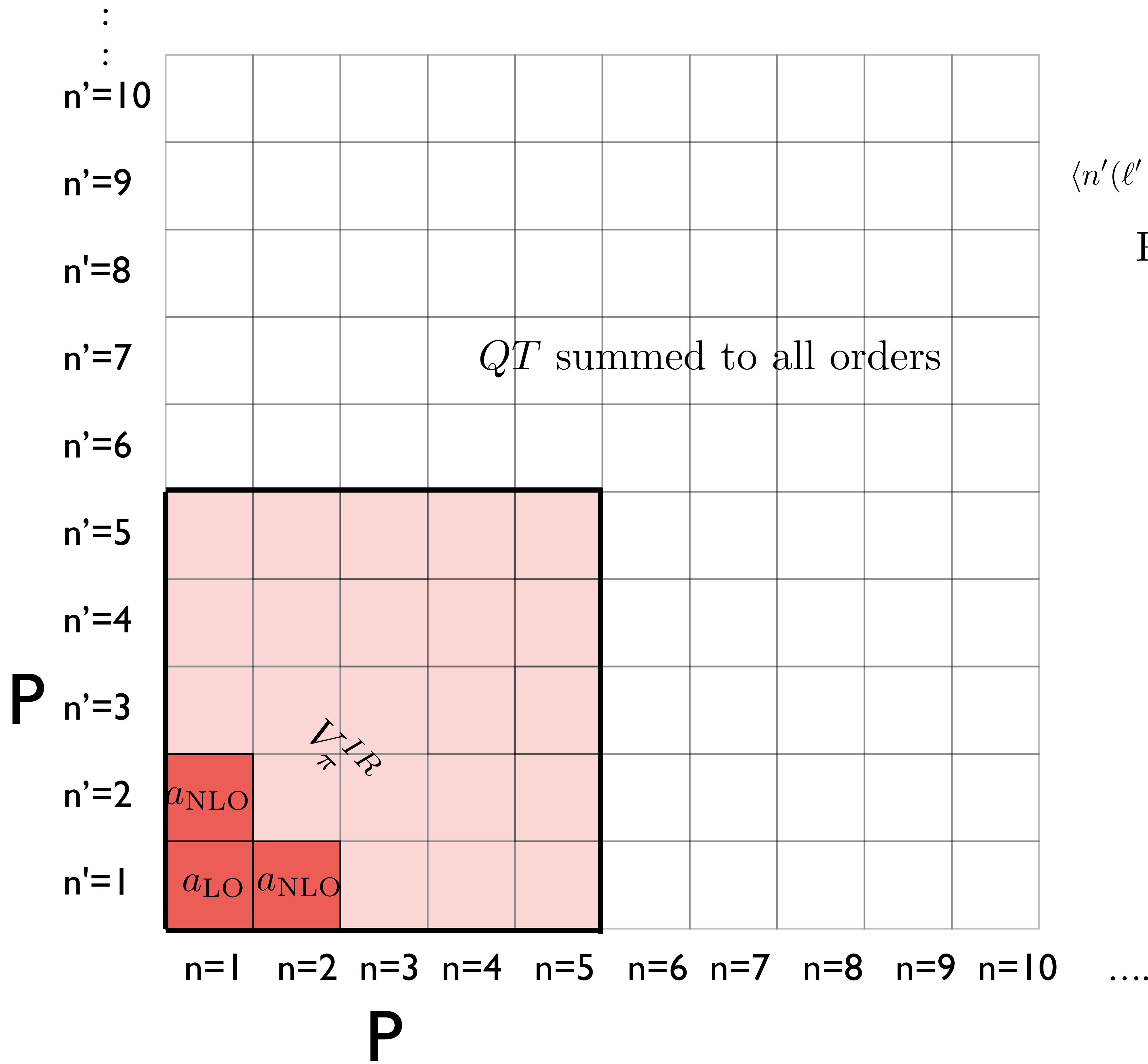


$\Lambda = 8\hbar\omega$

$\langle n'(\ell'=0S)J;T|\ H^{\rm eff}\ |n(\ell=0S)J;T\rangle$

Pionful HOBET LO

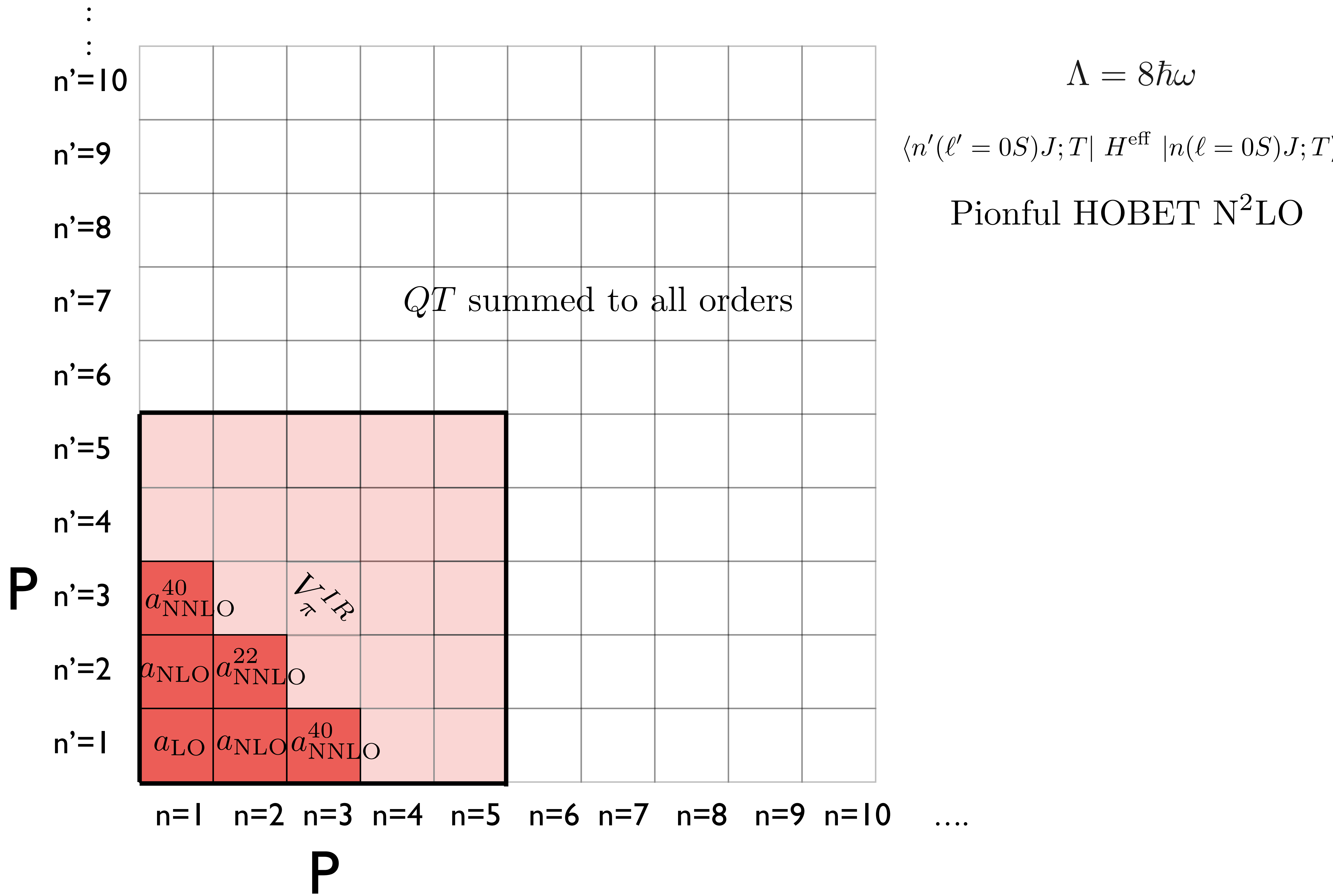
minimal self-consistent theory



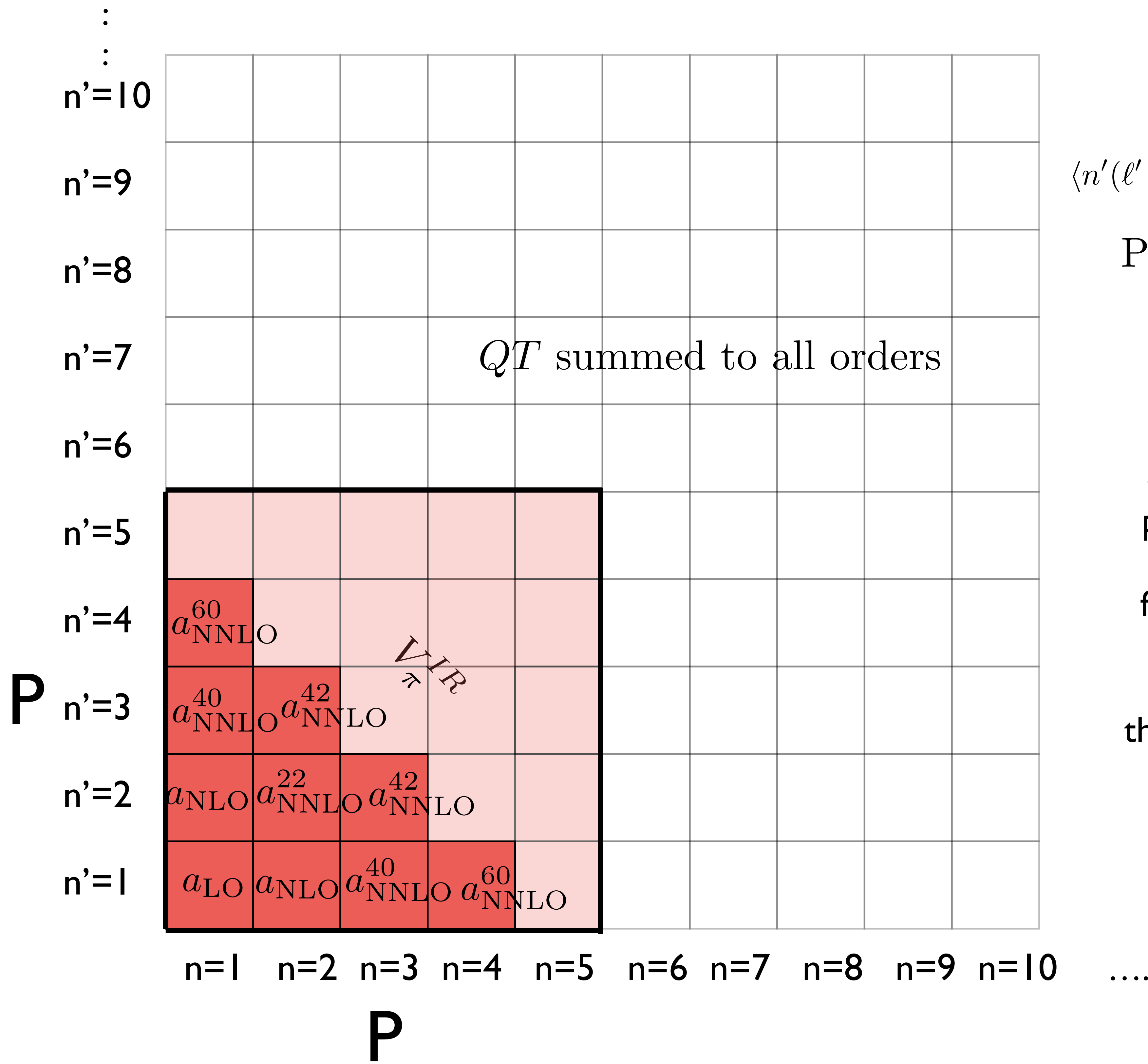
$\Lambda = 8\hbar\omega$

$\langle n'(\ell' = 0S)J;T|\ H^{\mathrm{eff}}\ |n(\ell = 0S)J;T\rangle$

Pionful HOBET NLO



$\Lambda = 8\hbar\omega$
$\langle n'(\ell'=0S)J;T|\ H^{\rm eff}\ |n(\ell=0S)J;T\rangle$
Pionful HOBET N^{2}LO
QT summed to all orders
n'=10
n'=9
n'=8
n'=7
n'=6
n'=5
n'=4
n'=3
n'=2
n'=1
P
$a^{40}_{\rm NNLO}$
V^{IR}_{π}
$a_{\rm NLO}$ $a^{22}_{\rm NNLO}$
$a_{\rm LO}$ $a_{\rm NLO}$ $a^{40}_{\rm NNLO}$
n=1 n=2 n=3 n=4 n=5 n=6 n=7 n=8 n=9 n=10 ....
P



$\Lambda = 8\hbar\omega$

$\langle n'(\ell' = 0S)J;T|\ H^{\text{eff}}\ |n(\ell = 0S)J;T\rangle$

Pionful HOBET N^{3}LO

As the short-range expansion is continued, pionic contributions are pushed further and further into the infrared

—

the short-range expansion accounts for all short-range physics

e.g., $^1S_0$

- Leading near-IR pionic contribution is governed by

$$a_{N^4LO} \sim a^{\pi}_{N^4LO} \sim \int d^3r r^8 e^{-r^2} \frac{e^{-\alpha r}}{\alpha r} V_\pi(r) \qquad r_{12} \equiv \sqrt{2}\ b\ r \qquad \alpha \equiv \frac{\sqrt{2} m_\pi c^2 b}{\hbar c} \sim 1$$

Depends on a single dimensionless parameter α

- Short-range operator structure is based on the HO raising/lowering operator

$$\begin{aligned} V_\delta^S = \sum_{n'n} \big[& a_{LO} |n'0\rangle\langle n0| \\ & + a^S_{NLO} [\boldsymbol{a}^\dagger \odot \boldsymbol{a}^\dagger |n'0\rangle\langle n0| + |n'0\rangle\langle n0| \tilde{\boldsymbol{a}} \odot \tilde{\boldsymbol{a}}] \\ & + a^{S,22}_{NNLO} \boldsymbol{a}^\dagger \odot \boldsymbol{a}^\dagger |n'0\rangle\langle n0| \tilde{\boldsymbol{a}} \odot \tilde{\boldsymbol{a}} \\ & + a^{S,40}_{NLO} [(\boldsymbol{a}^\dagger \odot \boldsymbol{a}^\dagger)^2 |n'0\rangle\langle n0| + |n'0\rangle\langle n0| (\tilde{\boldsymbol{a}} \odot \tilde{\boldsymbol{a}})^2] \\ & + a^{S,42}_{N^3LO} [(\boldsymbol{a}^\dagger \odot \boldsymbol{a}^\dagger)^2 |n'0\rangle\langle n0| \tilde{\boldsymbol{a}} \odot \tilde{\boldsymbol{a}} + \boldsymbol{a}^\dagger \odot \boldsymbol{a}^\dagger |n'0\rangle\langle n0| (\tilde{\boldsymbol{a}} \odot \tilde{\boldsymbol{a}})^2] \\ & + a^{S,60}_{N^3LO} [(\boldsymbol{a}^\dagger \odot \boldsymbol{a}^\dagger)^3 |n'0\rangle\langle n0| + |n'0\rangle\langle n0| (\tilde{\boldsymbol{a}} \odot \tilde{\boldsymbol{a}})^3] \big] \end{aligned}$$

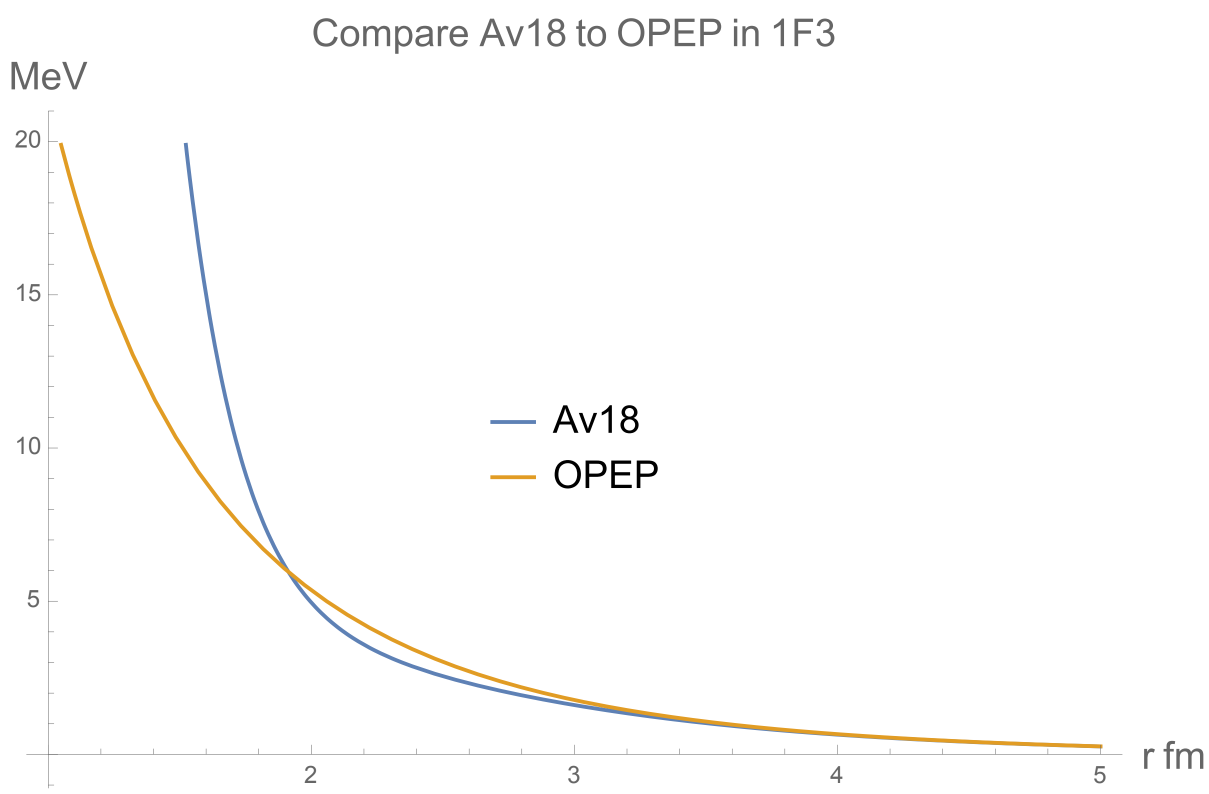
$$
\begin{aligned}
\langle n'(\ell'=0\ S)JM;TM_T|V_\delta^S|n(\ell=0\ S)JM;TM_T\rangle = d_{n'n}\big[a_{LO}^S - 2[(n'-1)+(n-1)]a_{NLO}^S \\
+4(n'-1)(n-1)a_{NNLO}^{S,22} + 4[(n'-1)(n'-2)+(n-1)(n-2)]a_{NNLO}^{S,40} \\
-8[(n'-1)(n'-2)(n-1)+(n'-1)(n-1)(n-2)]a_{N^3LO}^{S,42} \\
-8[(n'-1)(n'-2)(n'-3)+(n-1)(n-2)(n-3)]a_{N^3LO}^{60,S}\big]
\end{aligned}
$$

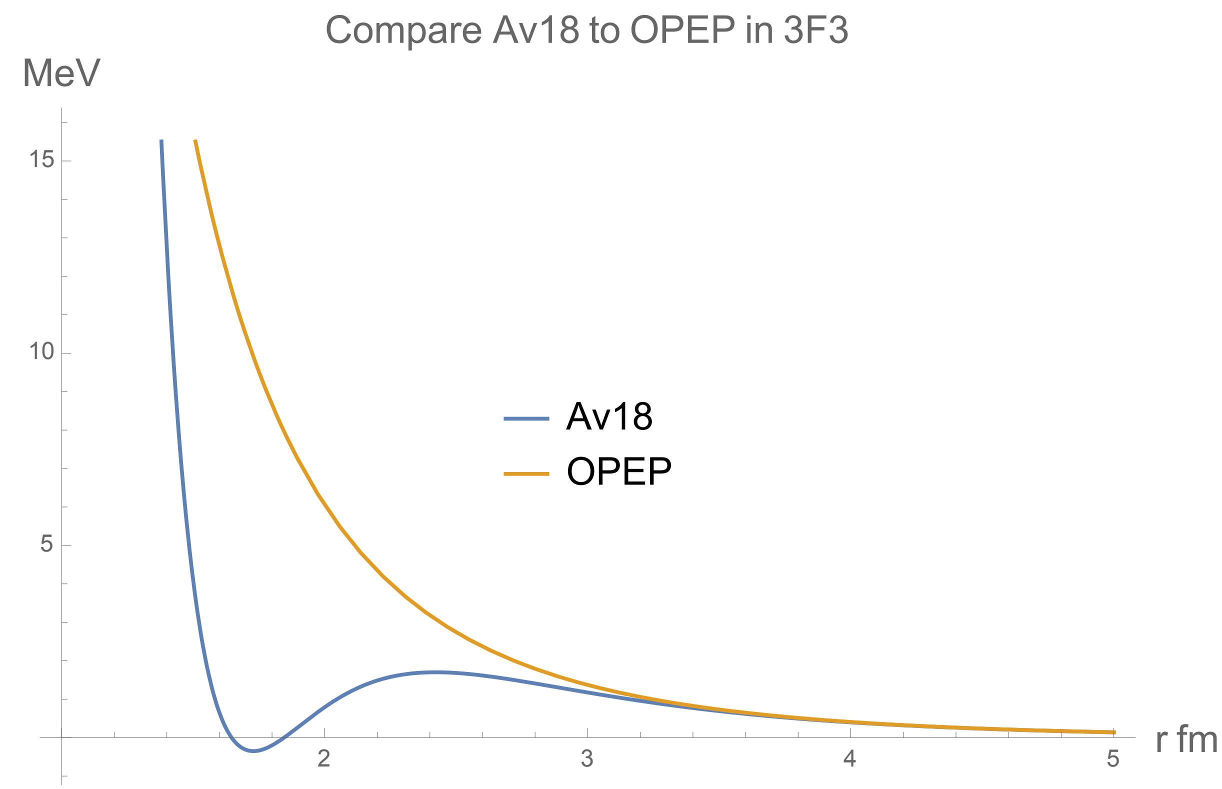
so $n'=1 \leftrightarrow n=1$ only gets a contribution from a_{LO}

and $n'=1 \leftrightarrow n=2$ gets contribution from a_{LO}, a_{NLO}

so scheme-independent fitting procedure

more generally, the lowest-energy information determines the LECs





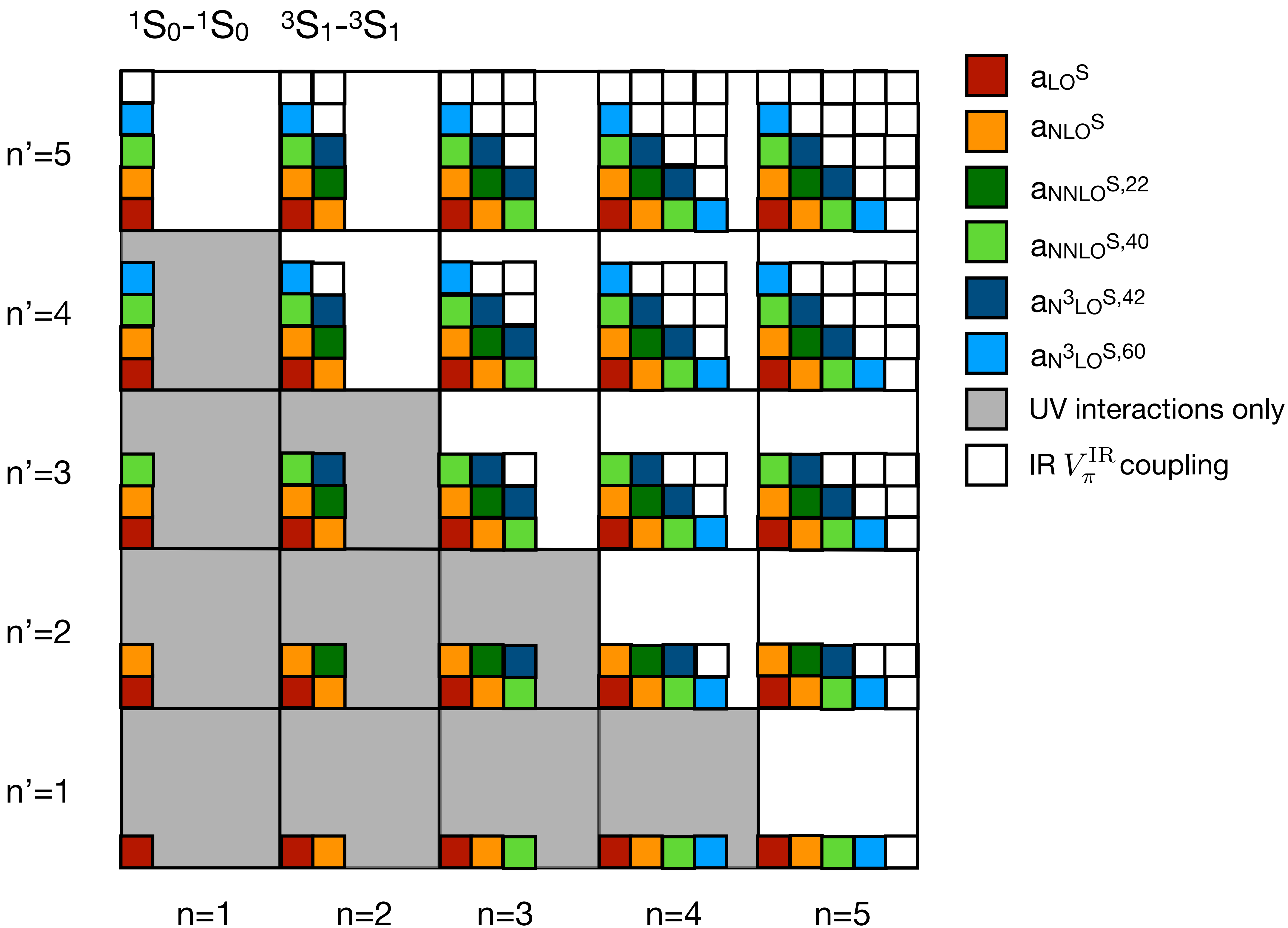
counting in singlet and triplet channels, pionful theory

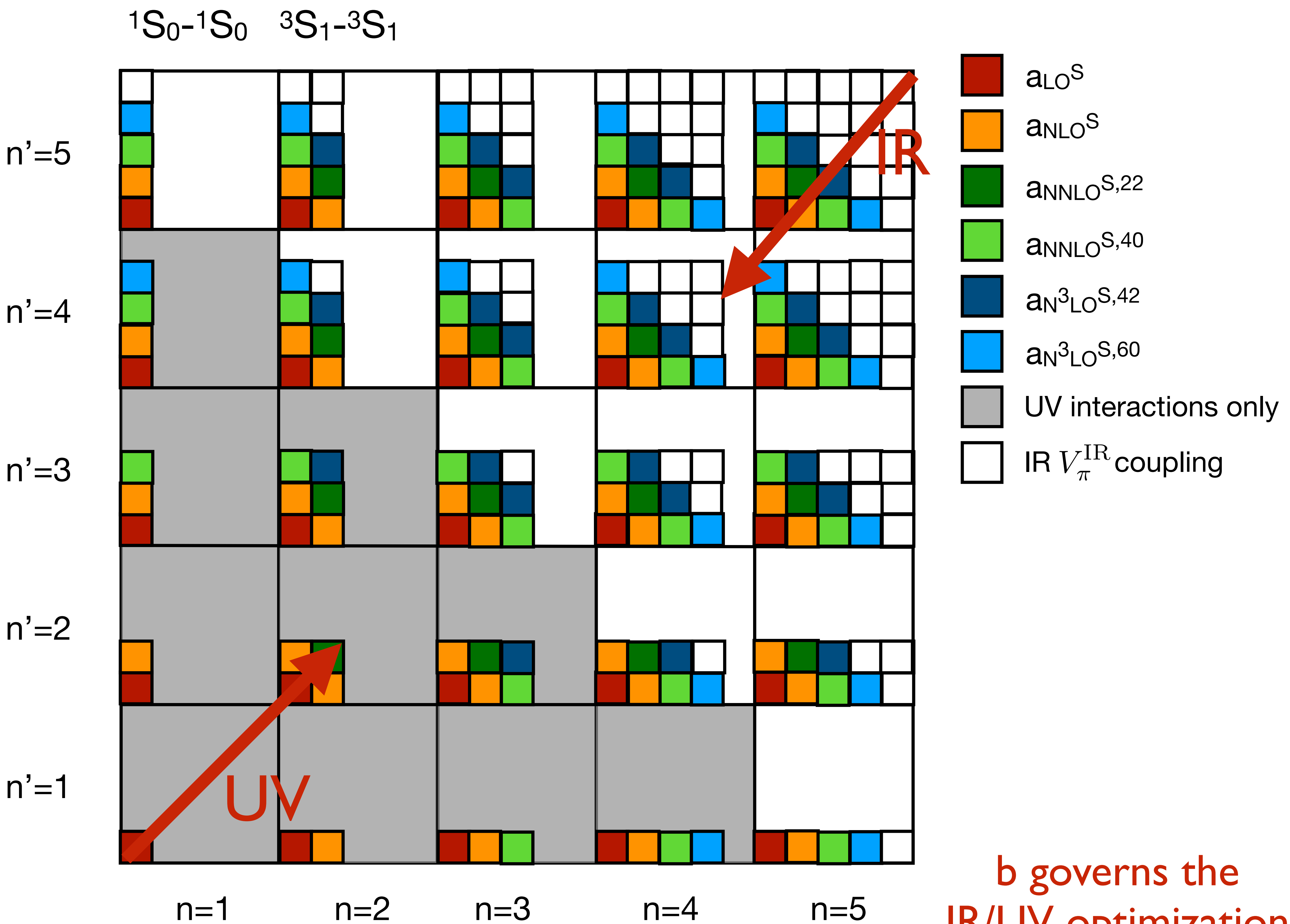
implicit dimensionless parameter $\left(\frac{a_{SR}}{b}\right)^2$ singlet $a_{SR} \sim 0.39f$

triplet $a_{SR} \sim 0.75f$

so typically an order of magnitude improvement per order

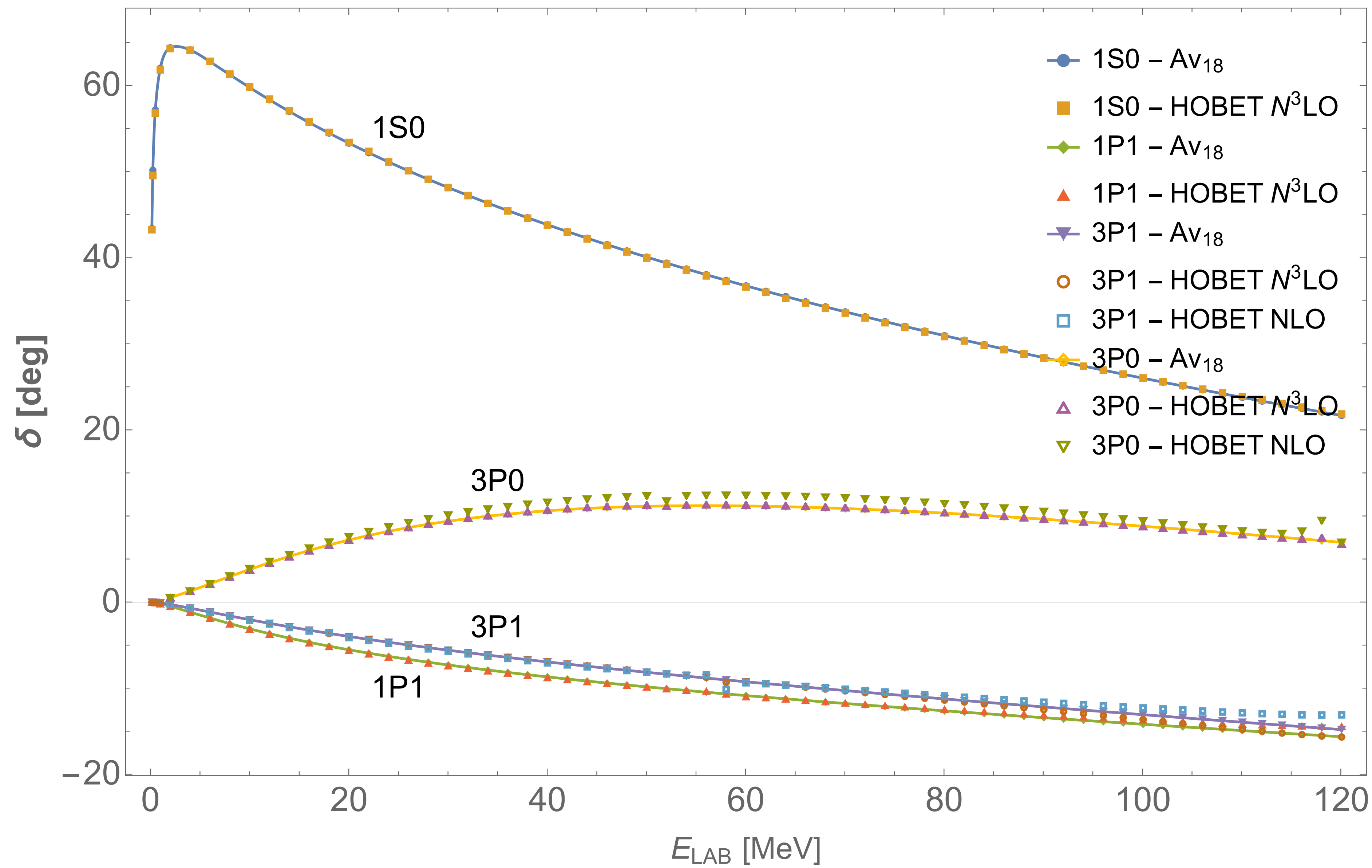
Chiral symmetry determines only the long-range physics - the white boxes





b governs the
IR/UV optimization

LECs determined directly from phase shifts (the information that used to go in NN potentials)



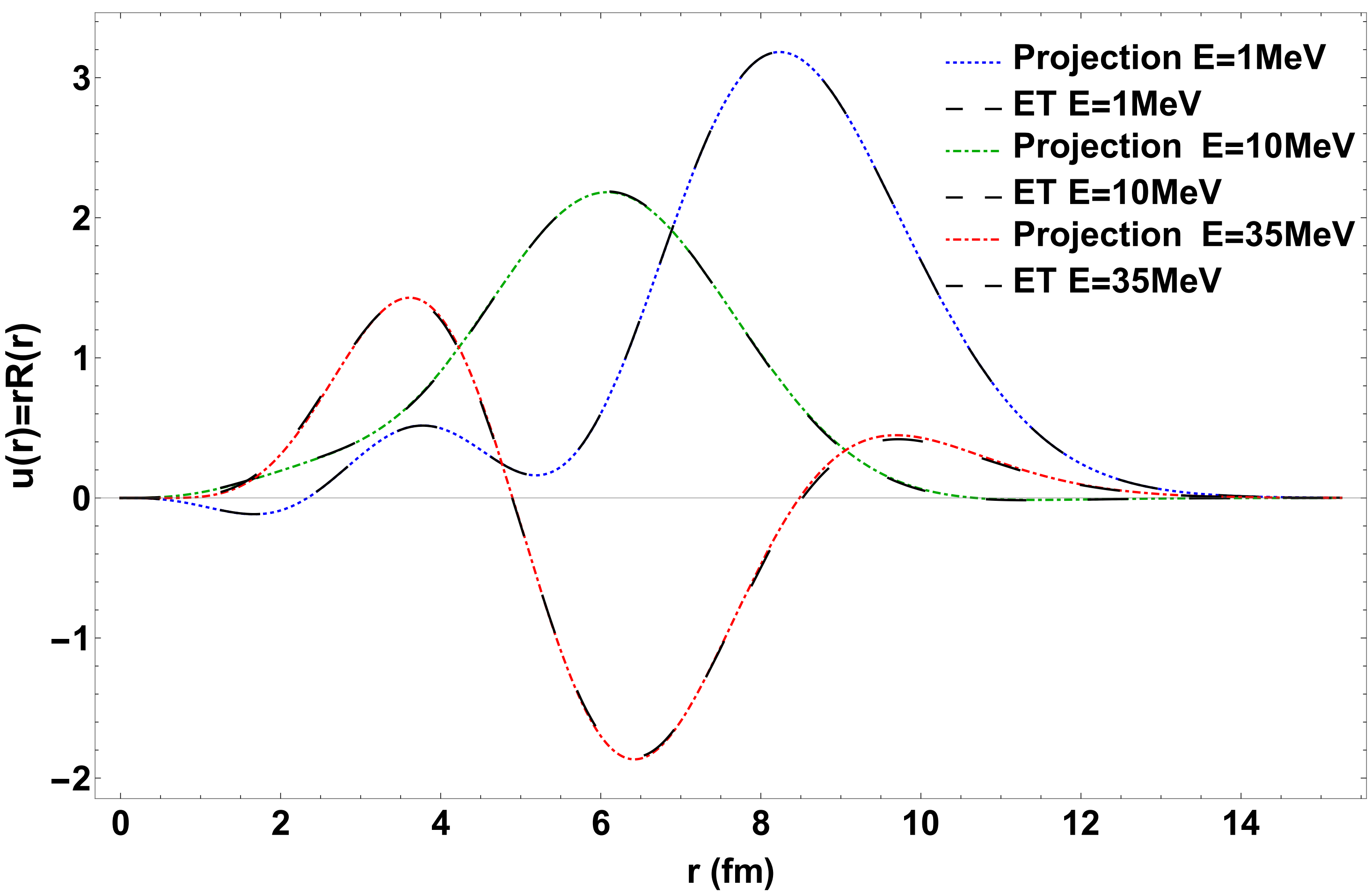
Described last meeting

Fig. 4. Phase shifts regenerated from LECs fit to data from 1 to 80 MeV and compared to the original phase shifts from Av_{18}. In the $^1S_0$ channel the low energy behavior down to 50 keV associated with a resonance at ~ 74 keV is reproduced from data above 1 MeV. In the $^3P_0$ and $^3P_1$ channels even NLO results based on a single LEC reproduce phase shifts quite well.

TABLE I. Deuteron channel: binding energy E_b as a function of the expansion order. Bare denotes a calculation with $T+V$
.

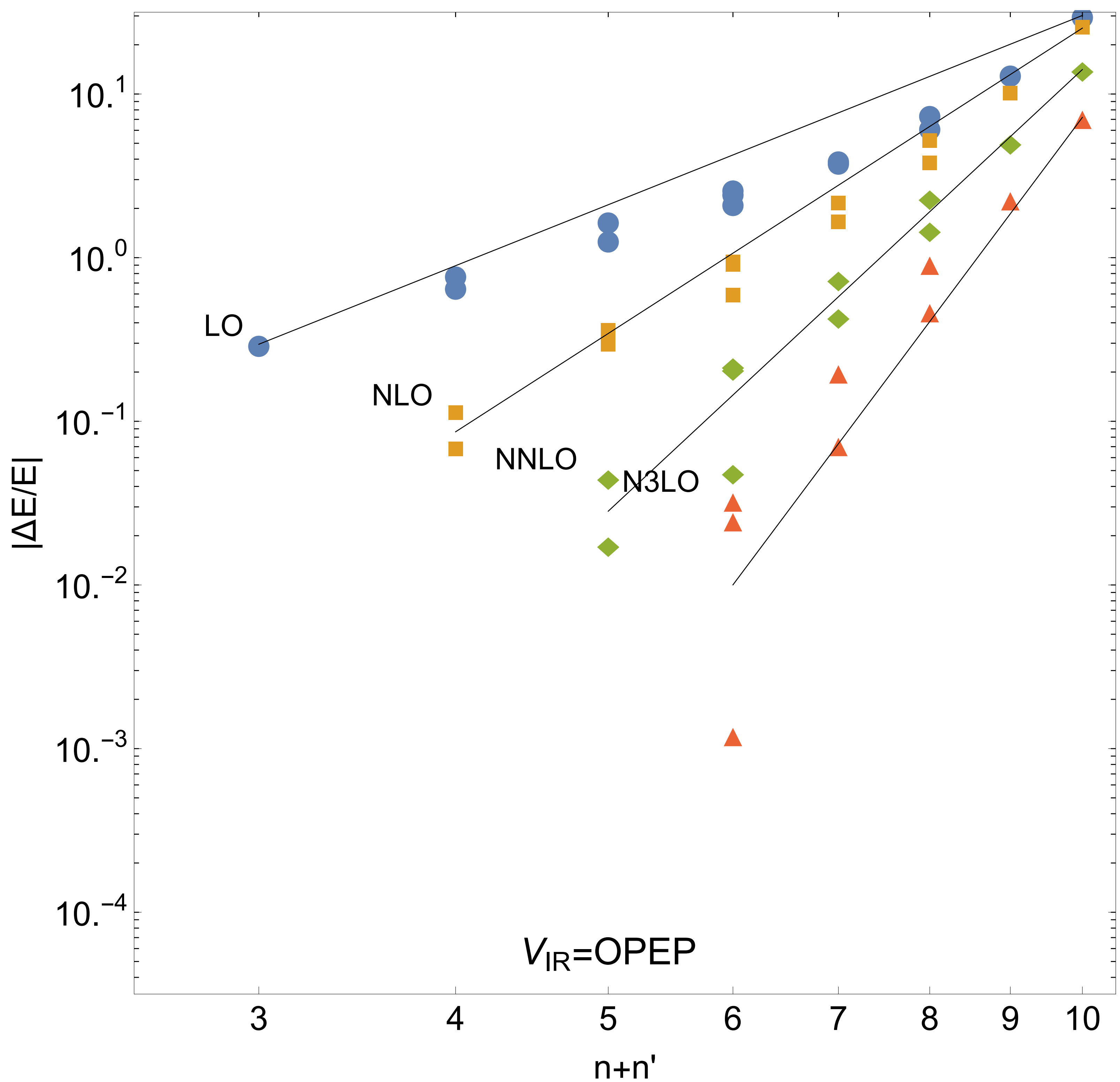
Order	$E_\text{b}^\text{pionless}$	C^2 $(LECs)$	$E_\text{b}^\text{pionful}$	C^2 $(LECs)$
bare	3.09525	-	-0.76775	-
LO	-1.27715	2.2E-2	-2.01110	1.9E-3
NLO	-1.95424	1.6E-2	-2.19833	2.2E-6
NNLO	-2.17307	6.7E-3	-2.21705	4.0E-8
N^3LO	-2.23175	1.3E-3	-2.22464	8.4E-9

Virtual perfect to the scattering data for E_{CM} 0-40 MeV: pionful HOBET accurate to 0.1 keV



|P| $P|\Psi\rangle$

Continuous function of E, r reproduced virtually exactly with 4 LECs

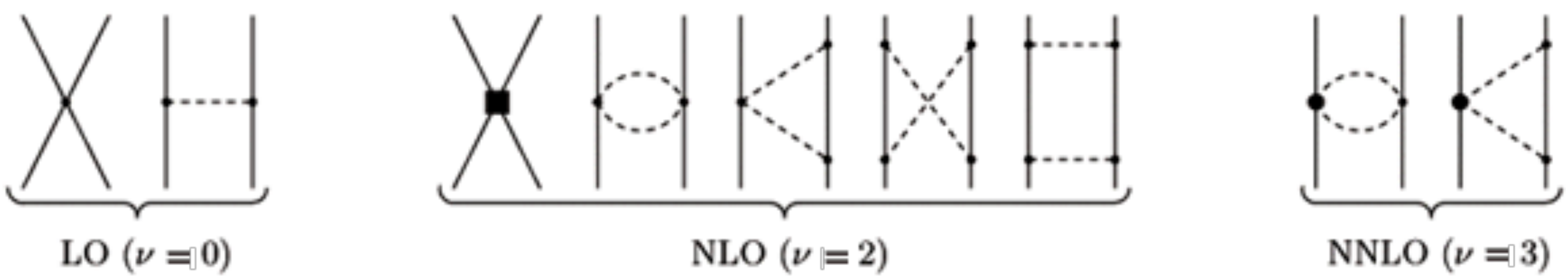


Converges rapidly: the improvement in interactions no used in the fit is systematic

Lepage Plot for Scheme-Independent Fitting: Pionful, Phase Shifts Only

HOBET's treatment of chiral symmetry: How efficient is the expansion?

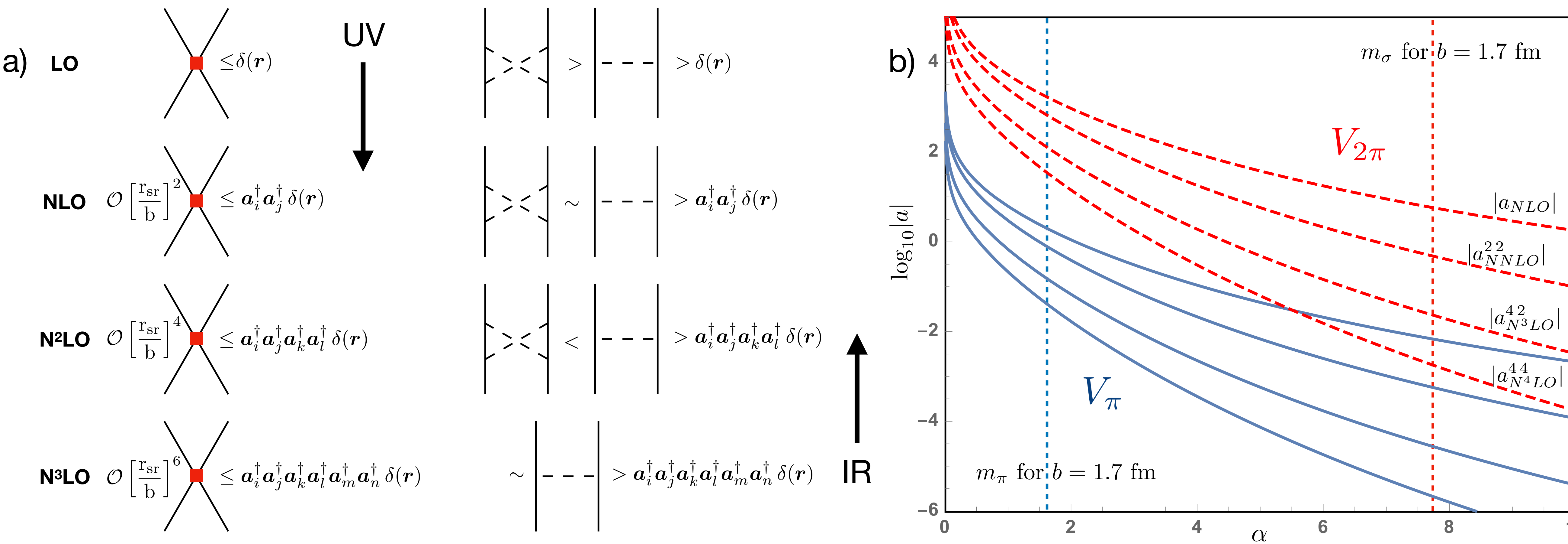
Standard chiral EFT includes the pion because of its dominance of the infrared, but treats it at all separations, leading to an increasing awkward problem as the under-lying short-range expansion is carried out, e.g.,



This is intuitively wrong: As the order (fidelity) of the short-range expansion improves, this should simplify - not complicate - the pionic mid-IR problem.

HOBET resolves this problem elegantly, as shown: the operator basis and its LECs exactly subtract out <u>all</u> short-range physics. A finite set of LECs for omitted high-order operators are taken from chiral symmetry.

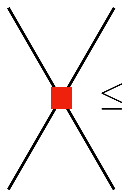
isoscalar sigma vs pion exchange: S-wave

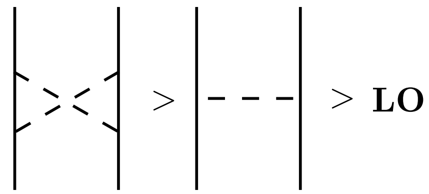


$|V_{2\pi}| > |V_\pi|$

everything above LO
treated as UV

a) **LO**





NLO

N²LO

N³LO

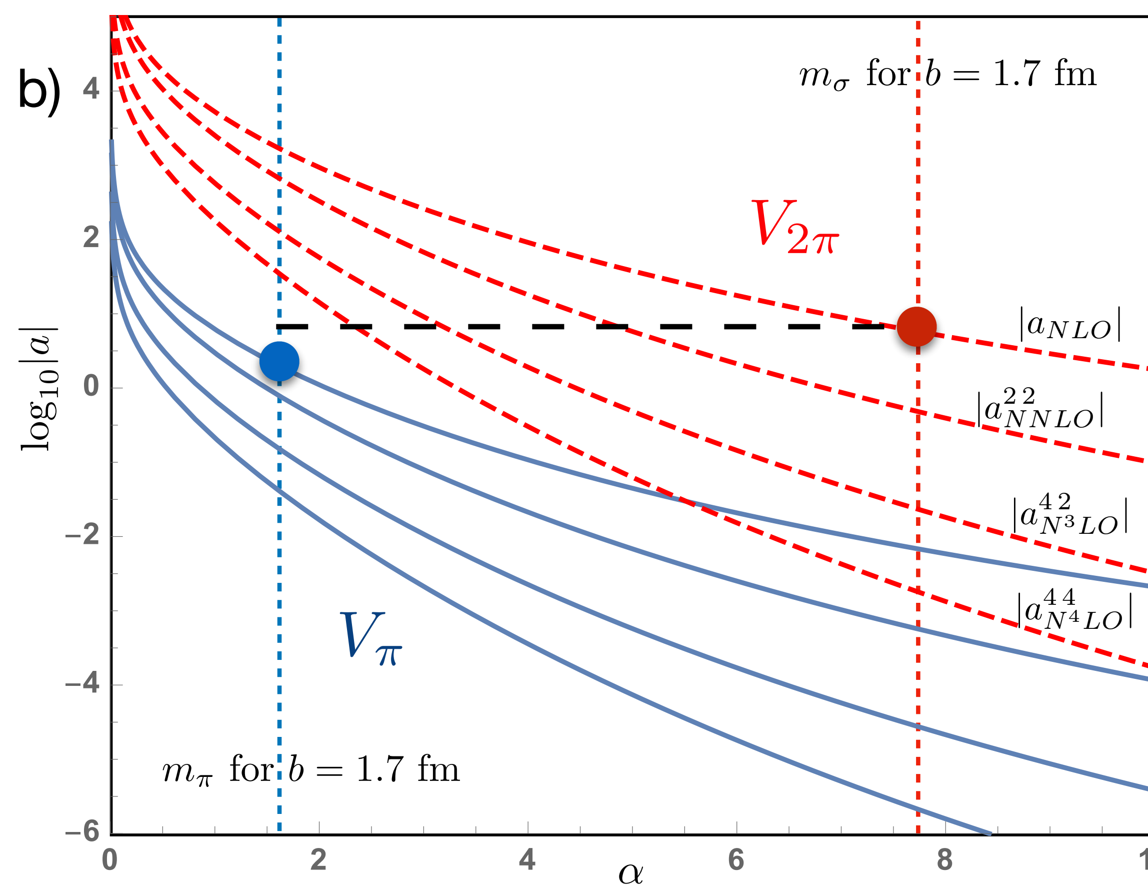
IR

<u>isoscalar sigma vs pion exchange</u>

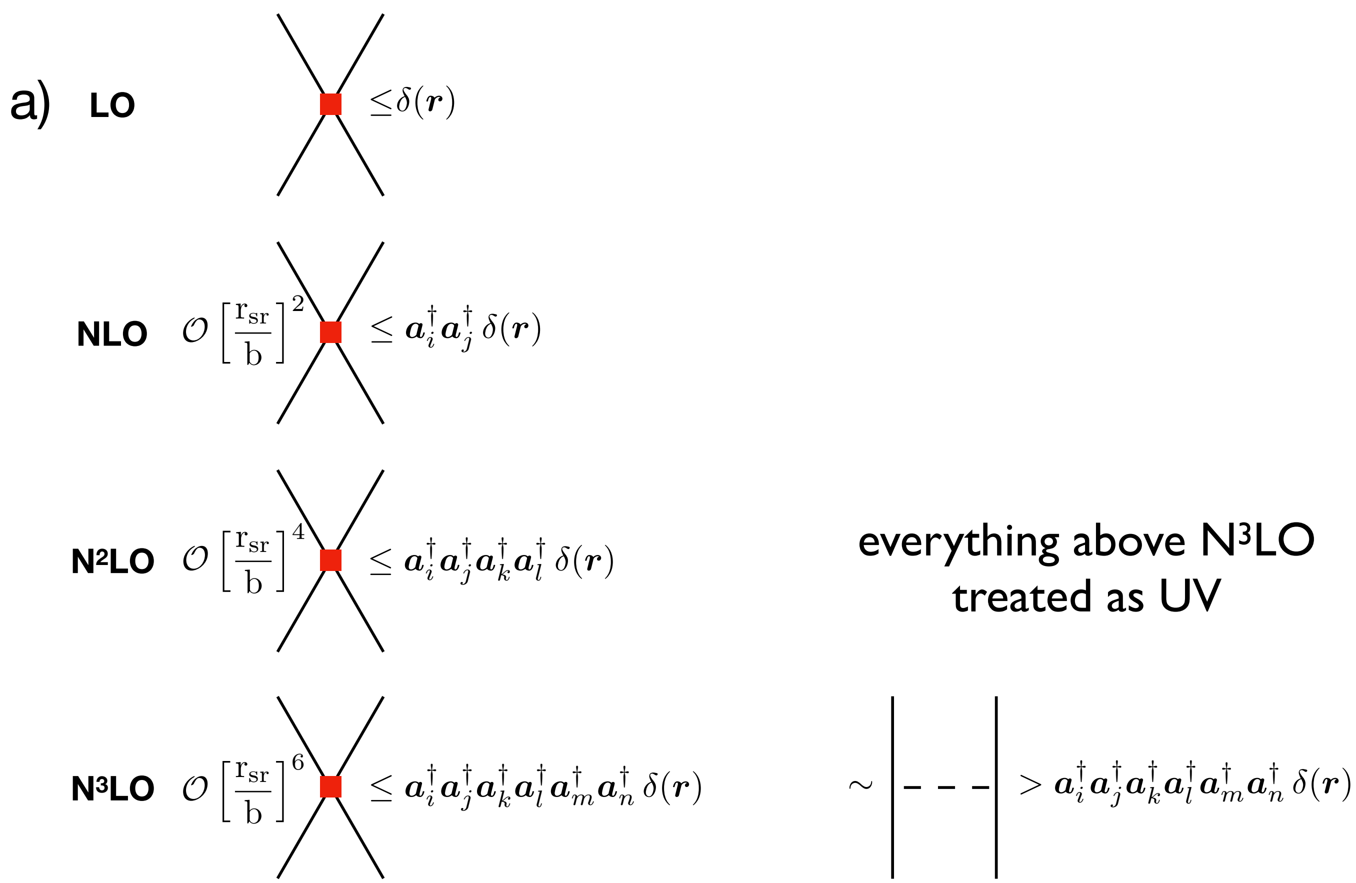
<u>LO Calculation</u>

NLO is the first contributing IR order

b)



$|V_{2\pi}| > |V_\pi|$

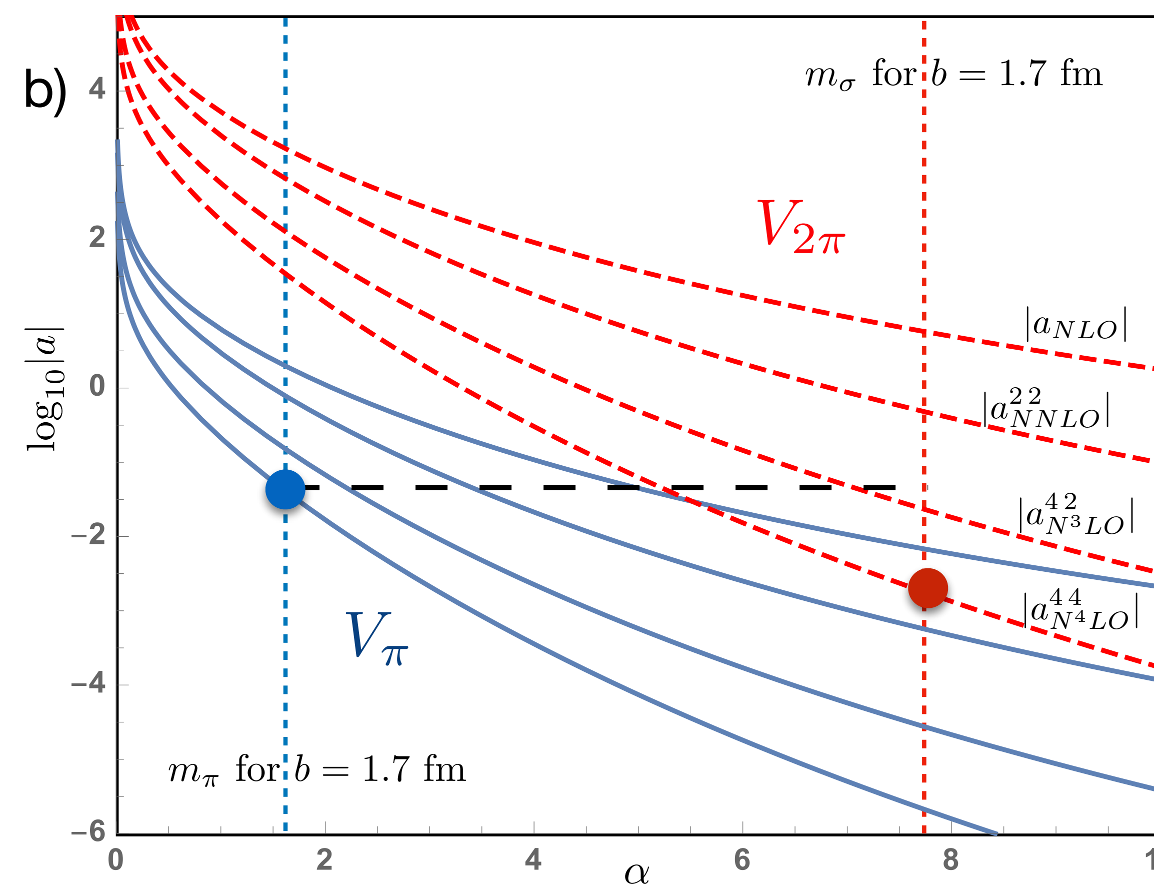


this is the behavior an efficient EFT should have

isoscalar sigma vs pion exchange

N3LO Calculation

N4LO is the first contributing IR order



$$|V_\pi| >> |V_{2\pi}| \sim \frac{1}{20}|V_\pi|$$

Final point: If we have the two-body P-space interaction $H_{12} = \; P_{12}(\Lambda)\; H_{12}^{\rm eff}(\Lambda)\; P_{12}(\Lambda)$

what is the solution of the three-body problem at the two-body level?

Translational invariance requires use of $P_{123}(\Lambda)$ where $\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3$

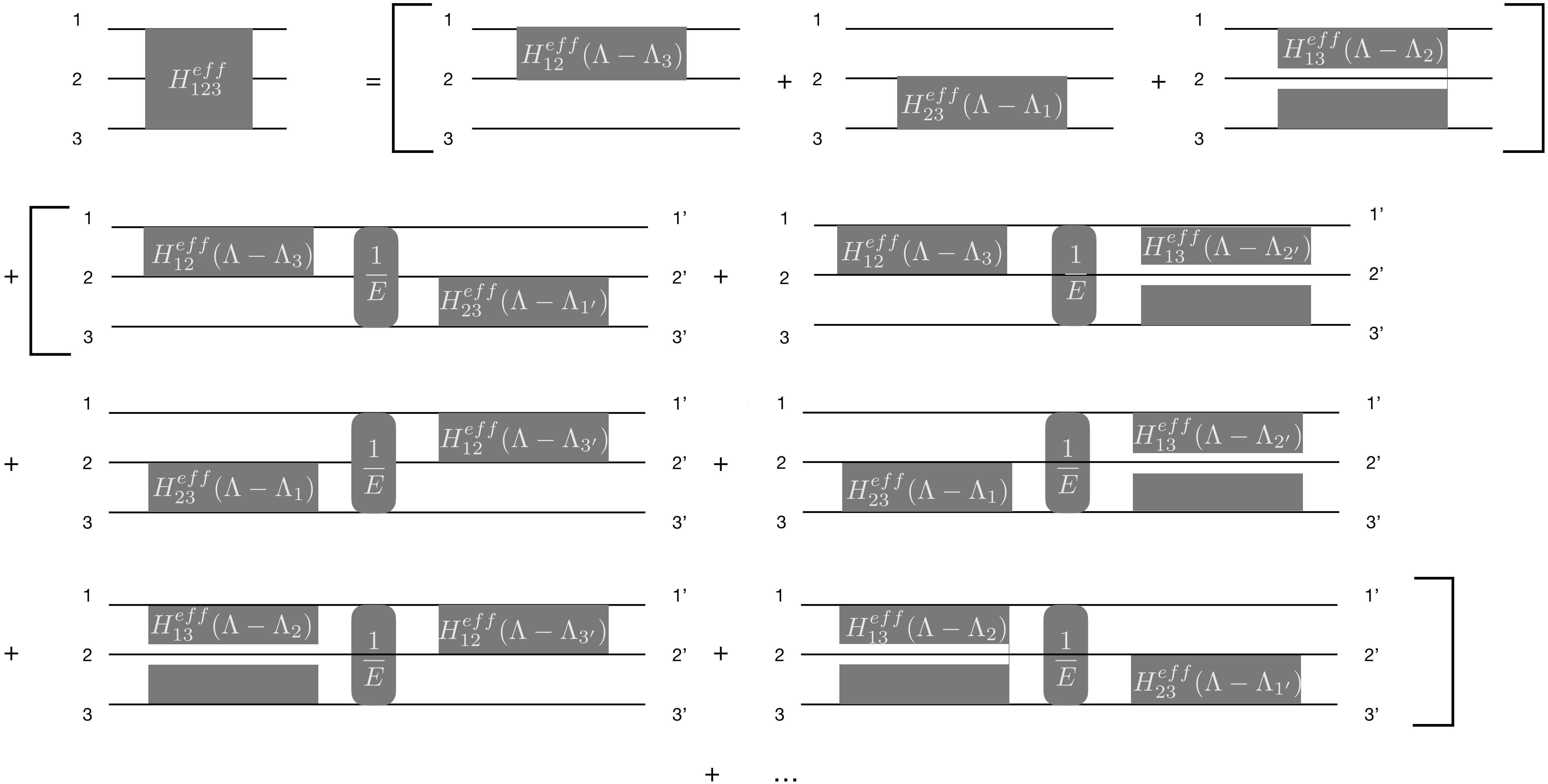
The BH equation is

$$P_{123}\left[H + H\frac{1}{E - Q_{123}H}Q_{123}H\right]P_{123}|\Psi\rangle = EP_{123}|\Psi\rangle$$

$$H = H_{12} + H_{23} + H_{13}$$

and when this is evaluated one answers the question

If I have the exact effective interaction at the two-body level for a HO Hilbert space, what is the corresponding form of the embedding of that effective interaction in a N-body system?



This series is simply summed: all matrices are finite, leading to a finite-basis Faddeev equation: this is what Ken is evaluating now.

This result is required by translational invariance - any simple embedding of a two-body interaction in a HO must take this form.