HOBET

I. Harmonic oscillator-based effective theory (HOBET)

II. Implementation

III. Phase shifts and Lepage plots

BERKELEY LAB

Office of Science
U.S. DEPARTMENT OF ENERGY

NSF

Berkeley
University of California

Goals of our HOBET Program

- Build a theory that is grounded in what we traditionally do in nonrelativistic NP that takes us from LQCD to the effective theory ("SM") in one step

- Base this theory on energy-dependent formulations of $H^{\rm eff}$, so that wave functions derived have a simple intuitive connect to exact results - projections

  Exactly describe the continuum (and bound states): infinite # of solutions

- Introduce chiral symmetry in a simpler way - as a long-distance correction, avoiding point-nucleon short-range pathologies that make EFT more difficult

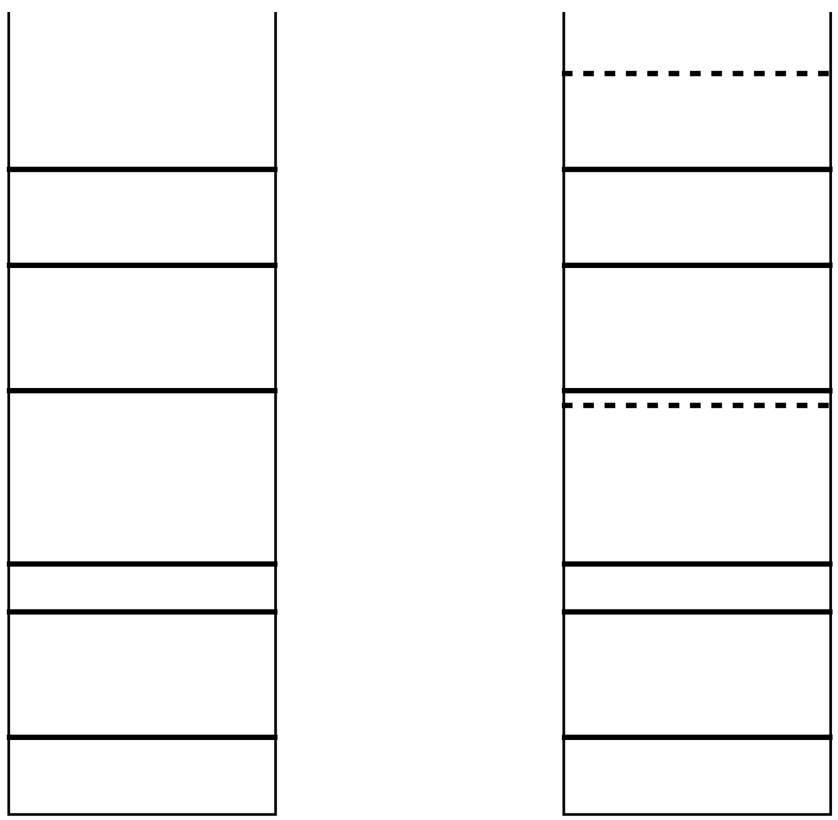
- Develop a version of HOBET to treat LQCD extrapolations to infinite volume

- Build a many-body theory based on a single testable approximation, a cluster expansion

Effective theory

- For decades we have thought about effective nuclear theory: the division of the Hilbert space into an included space P and an excluded space Q, and the determination of a H^{eff} in P (including models liked the SM)

- The first catastrophic failure of this theory was recognized in the early 1970s: perturbative efforts to generate H^{eff} derailed by intruder states



$4\hbar\omega$ intruder state

$$\langle 4\hbar\omega | H | 2\hbar\omega \rangle >> \Delta E$$

Nonperturbative corrections to H^{eff}

e.g., $P = 0 + 2\hbar\omega$

Shucan and Weidenmuller
Barrett and Kirson

There is now a major program based on a nonperturbative approach: ET

- Lots of wonderful things could be said about this approach … so it is admittedly unfair to focus on the negatives … but …

$E = \infty$ UV: QCD

$E = 1 \text{ GeV}$ NN potential

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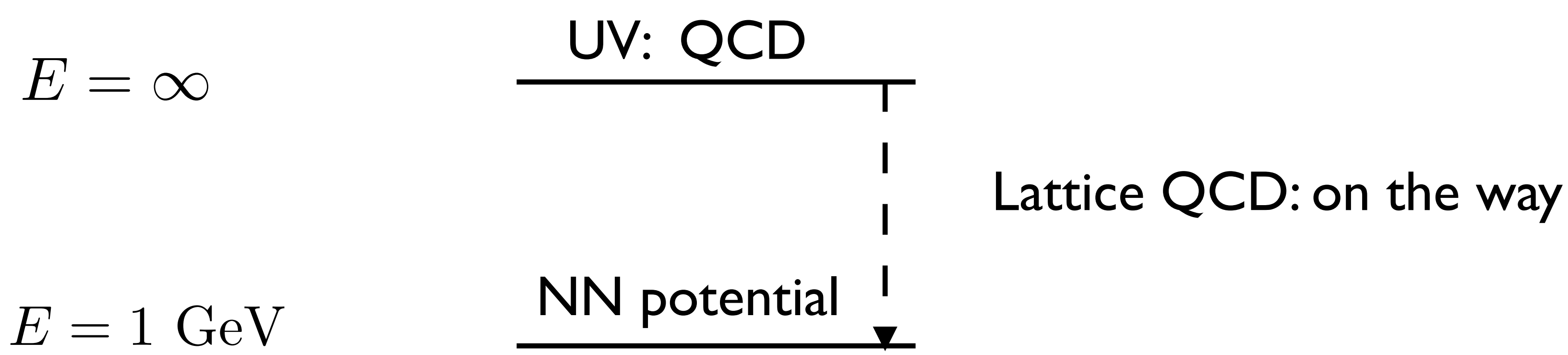

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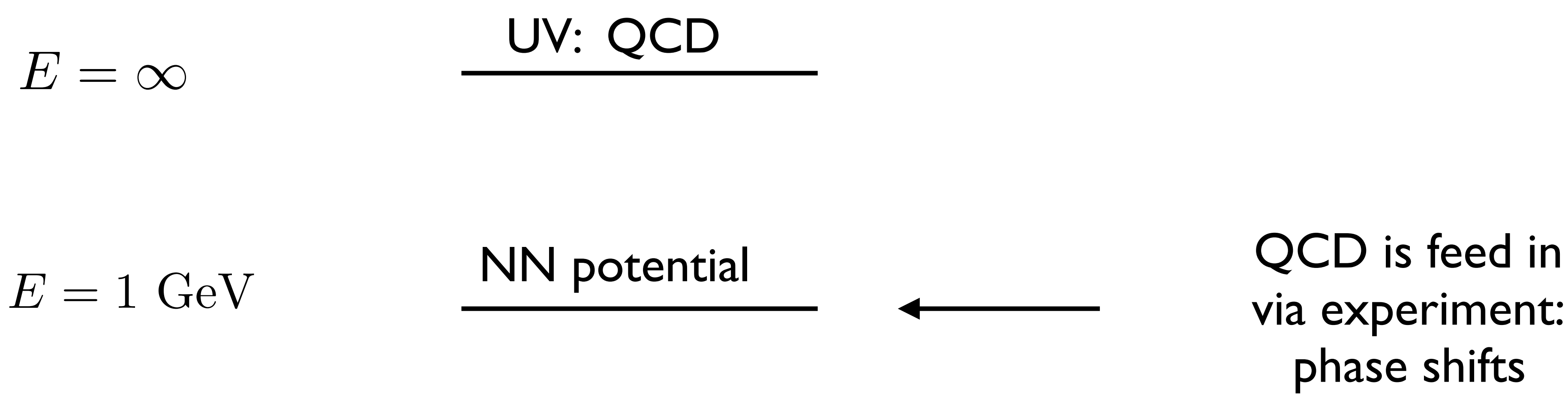

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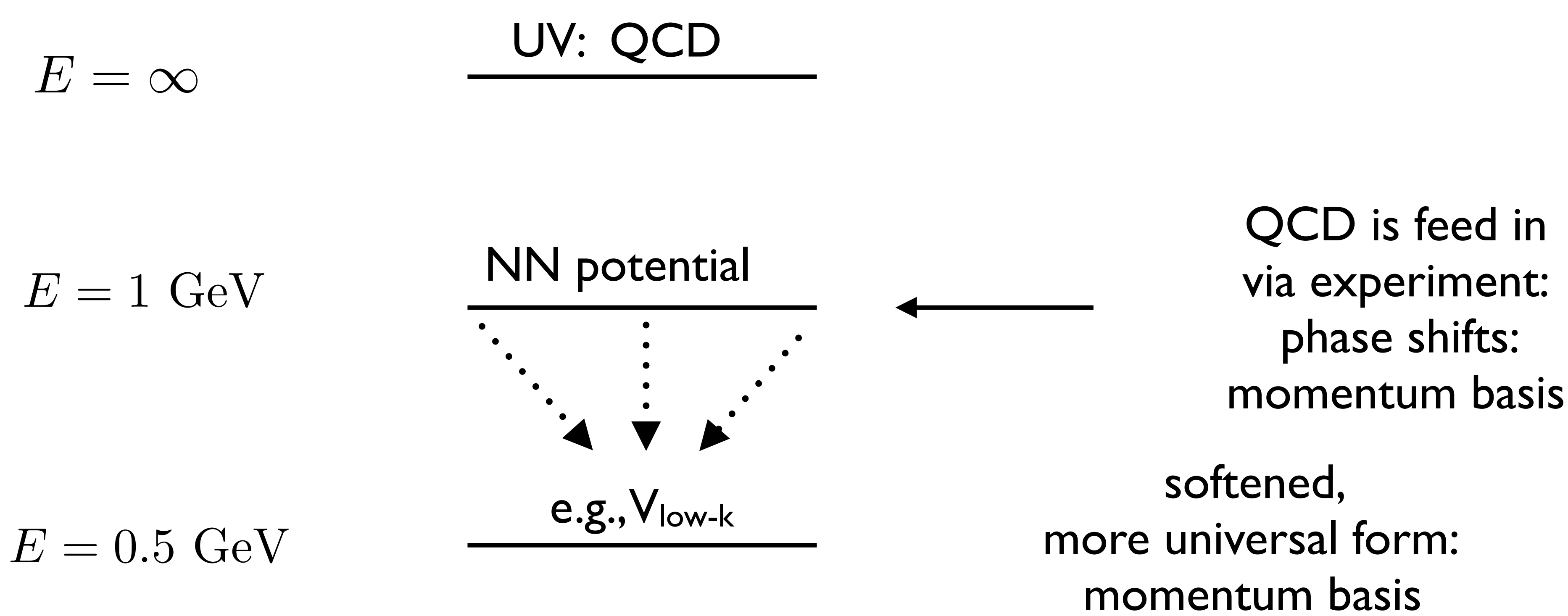

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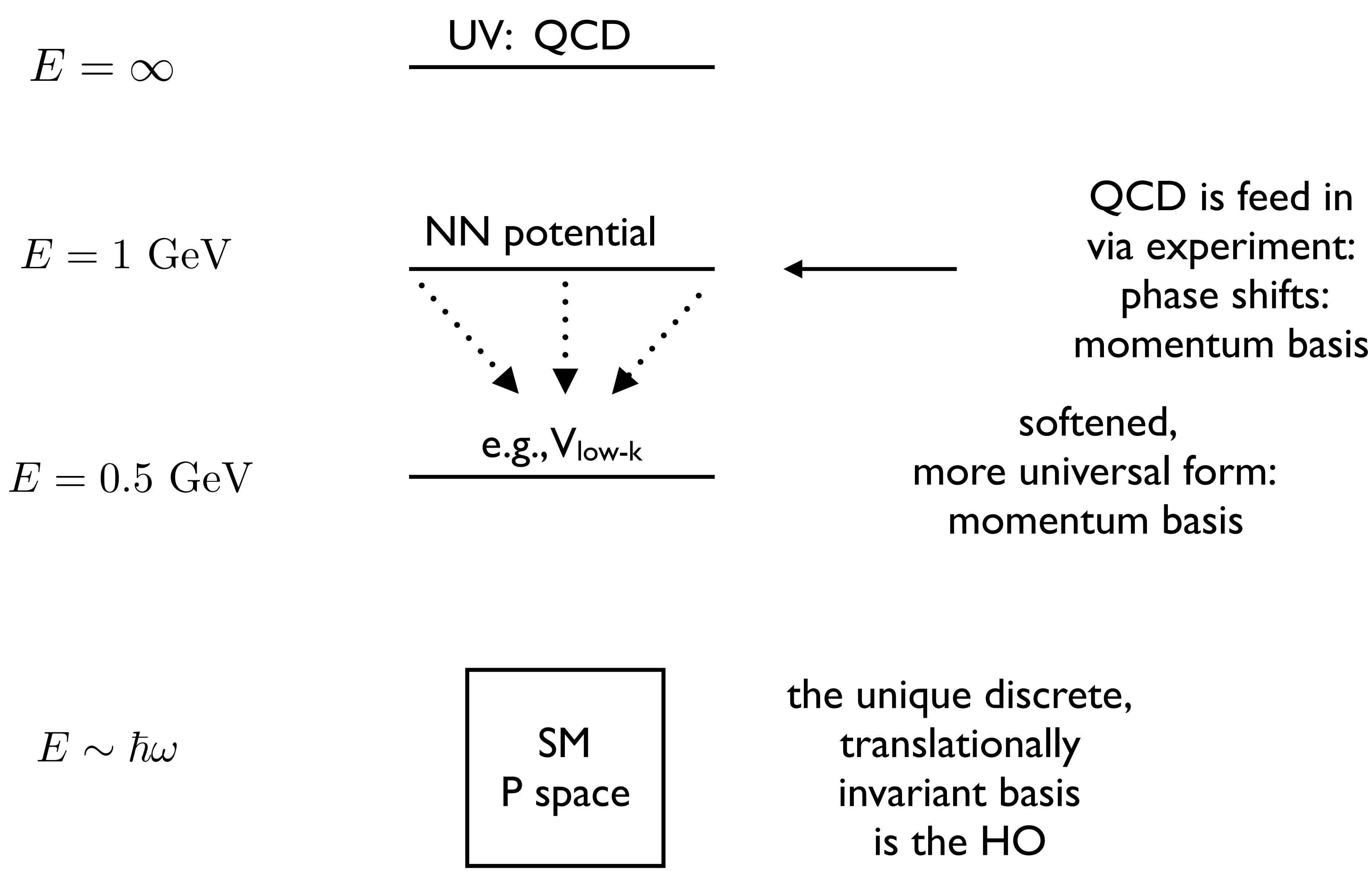

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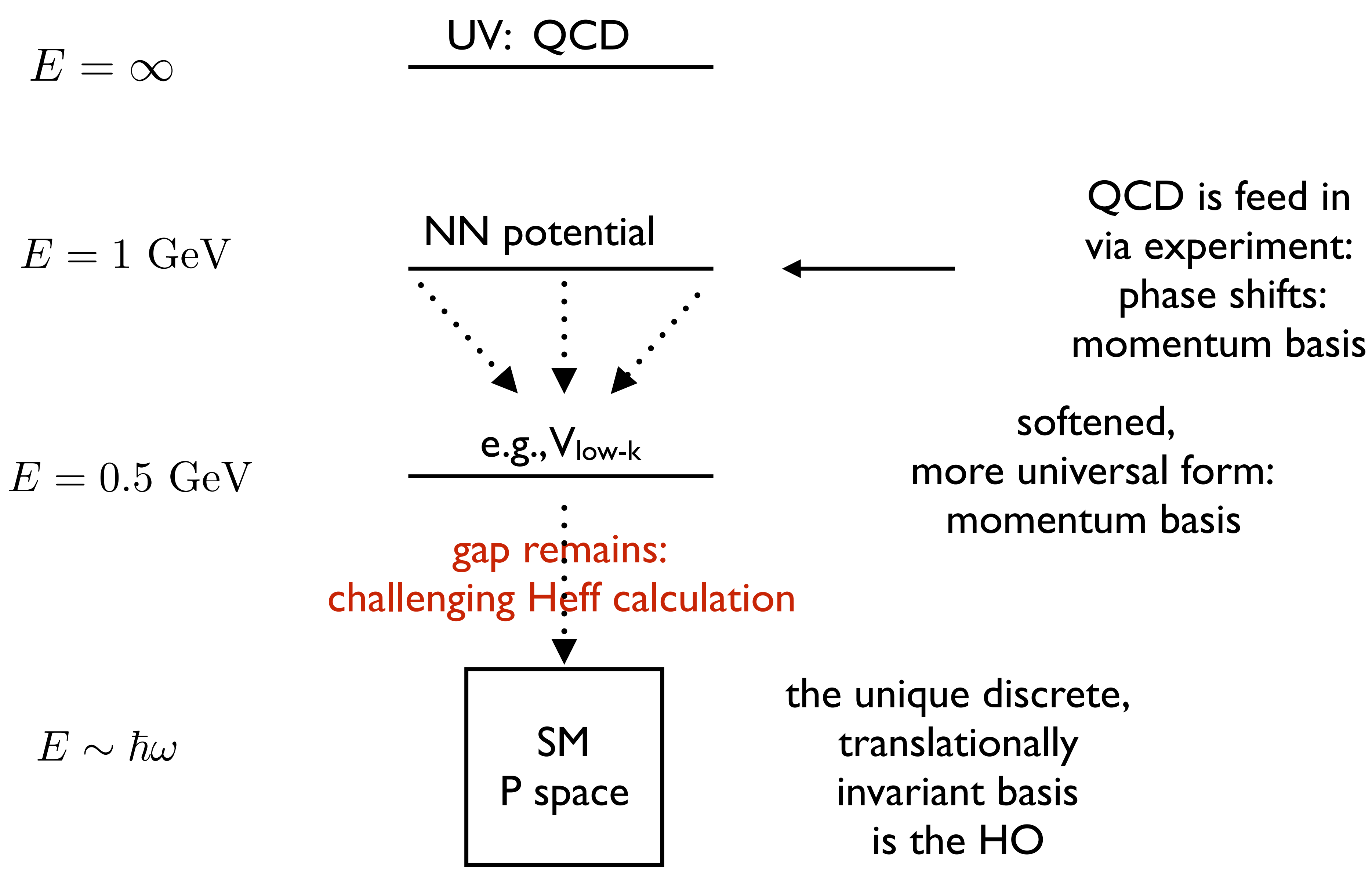
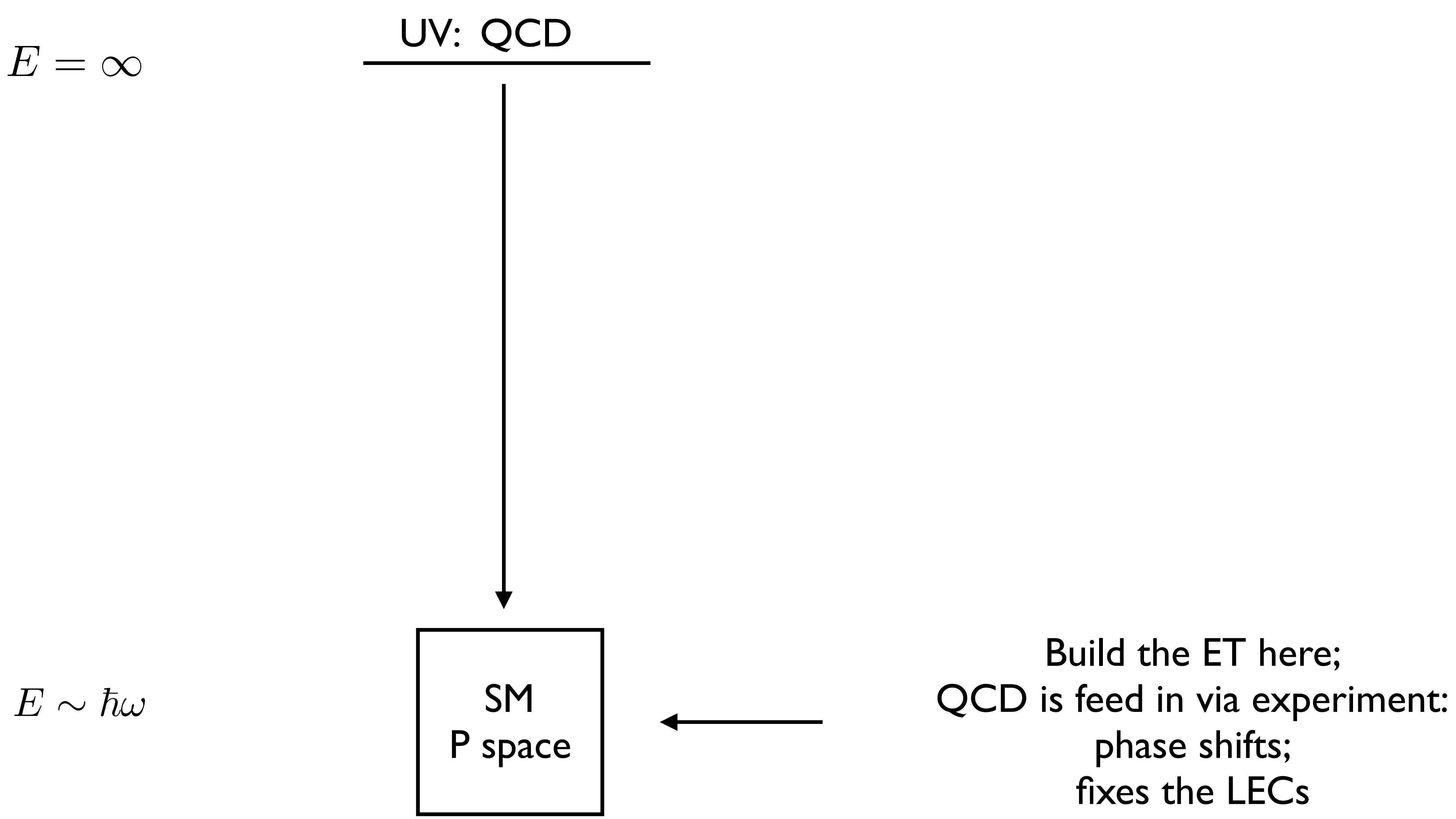
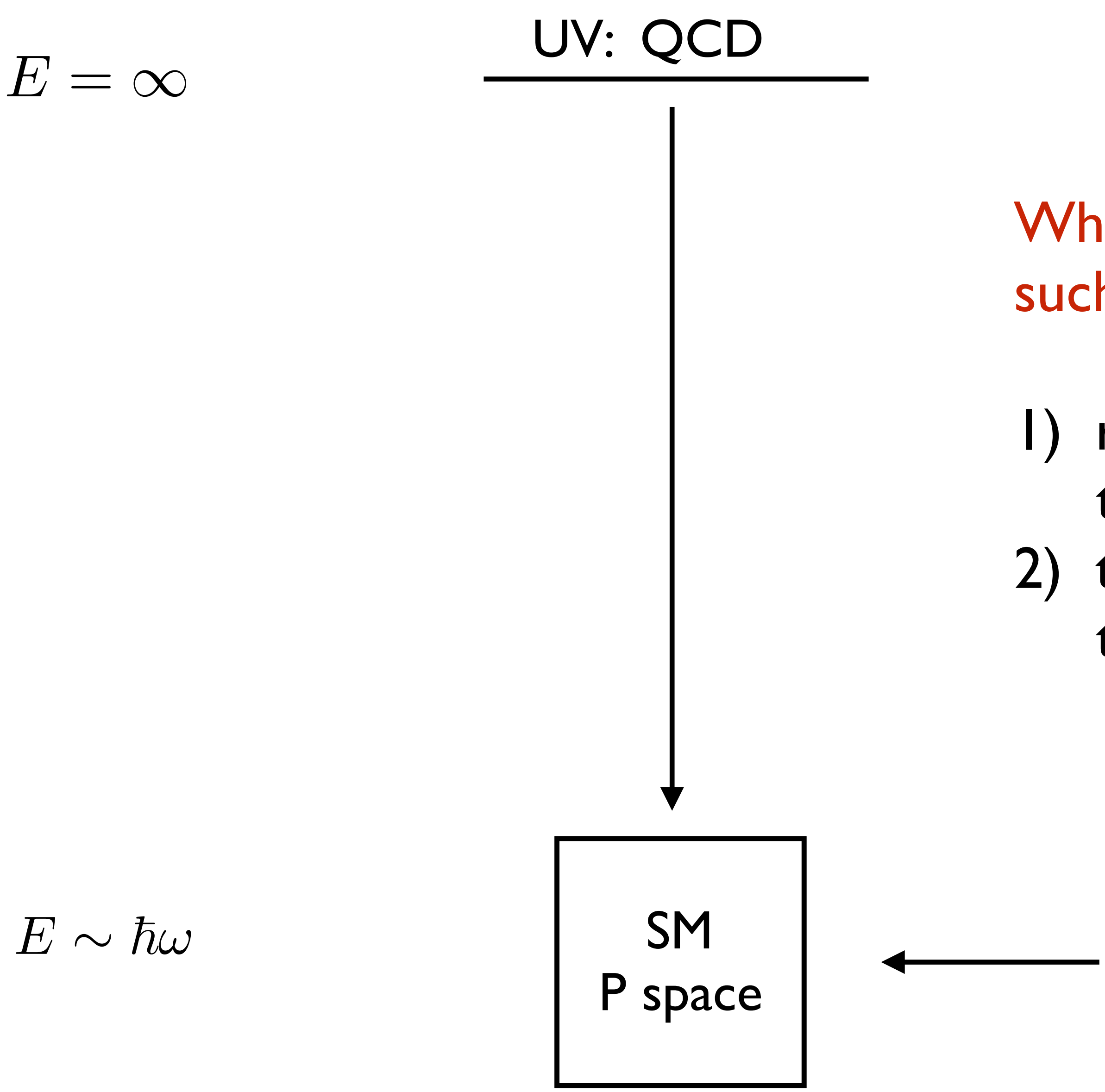


HOBET is based on three ideas

1) Simplify to a true ET



HOBET was design to simplify such procedures



What is the best starting point for building such a theory?

1) requires a theory that applies equally to bound states and continuum states
2) to use phase shifts, must have a theory that relates $H^{\rm eff}$ to $\delta(E)$

Build the ET here;
QCD is feed in via experiment:
phase shifts;
fixes the LECs

2) build a theory analytically continuous in energy (bound + continuum)

- Bloch Horowitz equation was developed in the 1950s: Often used in QM classrooms to derive Wigner-Brillouin perturbation theory

- In the present application we will divide the Hilbert space into P+Q, with a translationally invariant HO Slater determinant basis

$$P \rightarrow P(b, \Lambda)$$

P is the space and the cutoff scheme

- The BH equation in P gives the exact energy and exact $P|\Psi\rangle$

$$H^{eff} P|\Psi\rangle = EP|\Psi\rangle \qquad H^{eff} = P\left[H + H\frac{1}{E - QH}QH\right]P$$

Solve self-consistently

Properties:

- An infinite number of solutions, including essentially all of the continuum

- A simple effective wave function: the restriction of the true wave function to P

- No intruder states: every state mixing with P is generated

- Results independent of b, Λ (though good choices speed convergence)

ETs commonly formed are Hermitian, energy independent

- Ben Day around 1960: folded diagrams, nonHermitian, energy-independent interactions preserving good properties of the BH equation (nonorthogonality)

- But the field prefers to create Hermitian, energy-independent

- P has dimension D: one can find such an $H^{\rm eff}$ that (at most) reproduces D eigenvalues
  - Lee-Suzuki procedure: simple interpretation of the wave function lost

So why do folks make this choice?

- easy to fall into the trap of believing E-dependent $H^{\rm eff}$s are more complicated

Not so - you can have your cake, and eat it too

- An infinite number of solutions, including essentially all of the continuum

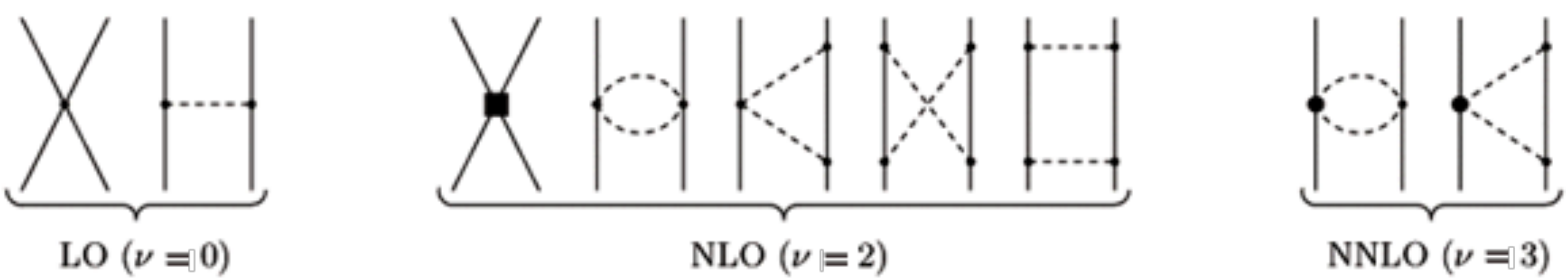
- A simple effective wave function

- No intruder states: every state mixing with P is generated

3) build in chiral symmetry in a more efficient way

- It is smart to build into an EFT our knowledge of the long-range potential - the light pion determines the tail

- But chiral interactions are introduced as pion exchanges between point nucleons:
  - introduces a tensor force with a $1/r^3$ short-range singularity
  - complicated to regulate as it scatters in Q
  - unphysical: nucleons are composite, and pions have nothing to do with the short-range NN potential



Life would be much simpler if there were a clean way to treat only the desired long-range contribution of the pion

These three ideas are the motivation for HOBET …

BH Formulation

- Nonrelativistic effective theory that is formulated in a HO P-space: discrete but translationally invariant

- Analytically continuous in E: applies equally to bound states or reactions

- Based on a reorganization of the Bloch-Horowitz equation (WH + Tom Luu). Here $E,\ |\Psi\rangle$ are the full solution

$$PH^{\rm eff}P|\Psi\rangle = EP|\Psi\rangle$$

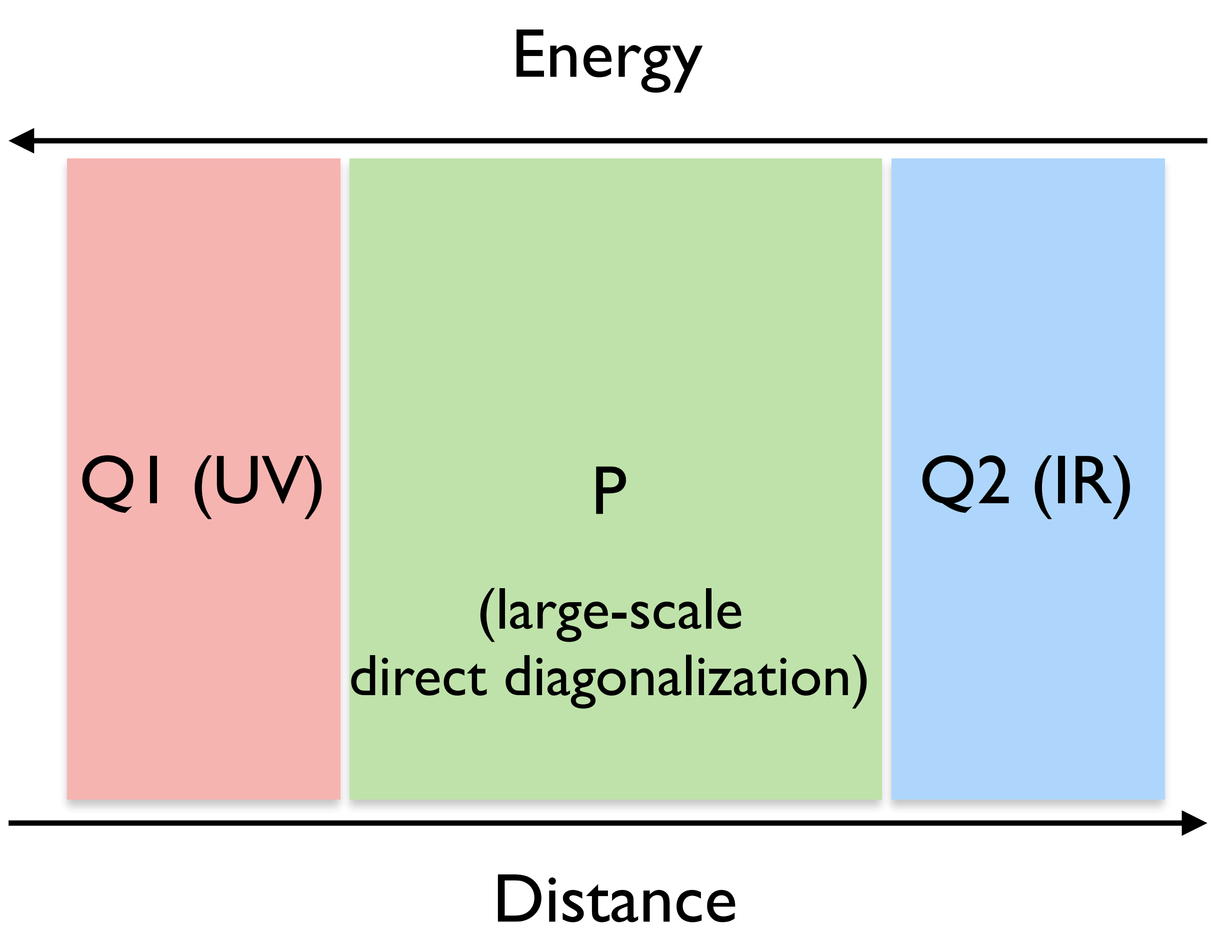
$$G_{QT} = \frac{E}{E-QT} \qquad G_{QH} = \frac{1}{E-QH}$$

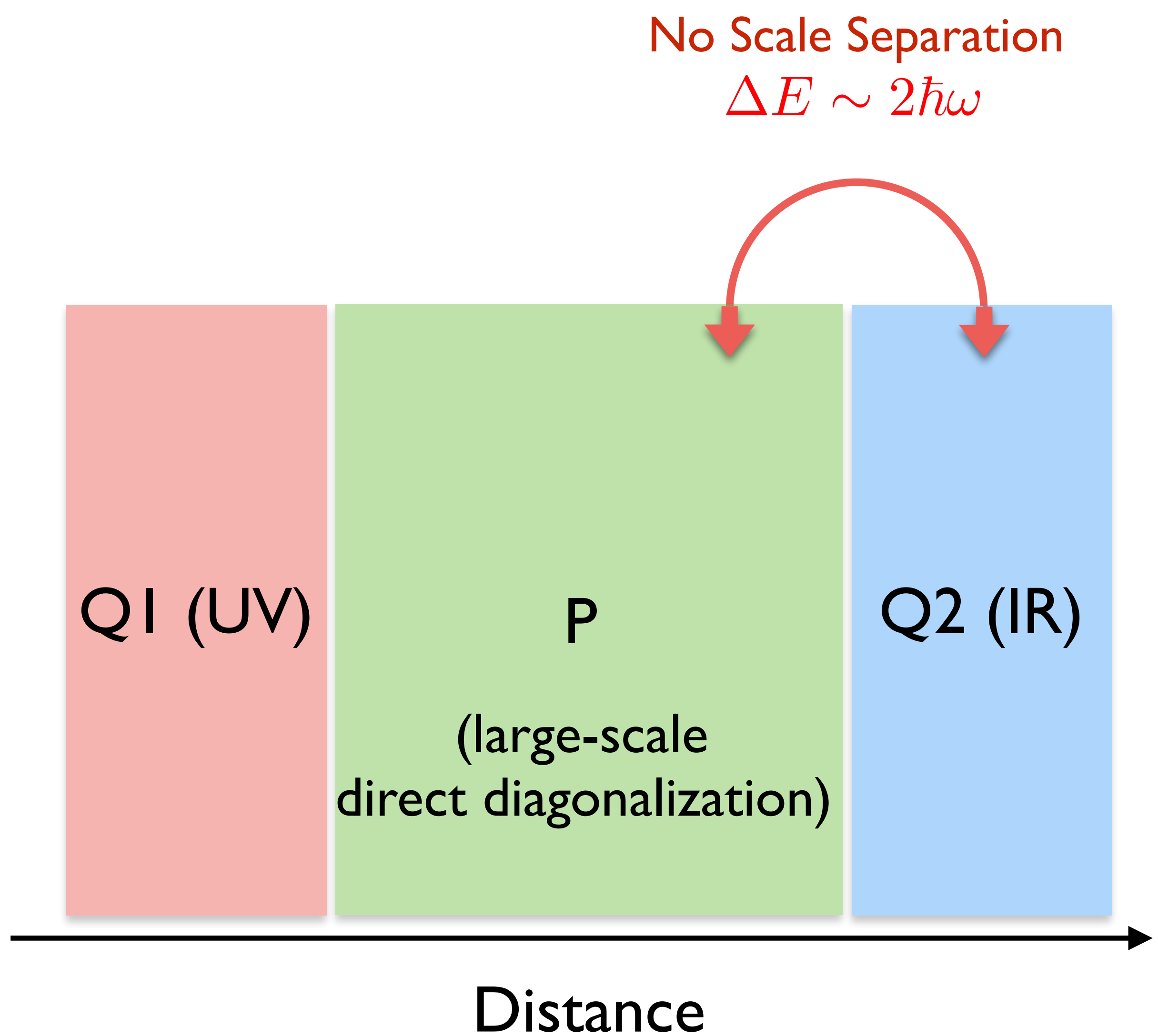
$$H^{\rm eff} = G_{TQ}\left[T + T\frac{Q}{E}T + V + VG_{QH}QV\right]G_{QT}$$

UV - P - IR Factorization

- Nuclear ground states are a compromise between the UV and the IR: kinetic energy is minimized by delocalization; potential energy is minimized by localizing at scales $\sim 1/m_\pi$

- Corrections due to omitted IR and UV physics are roughly comparable in importance — but differ greatly in their consequences for ET





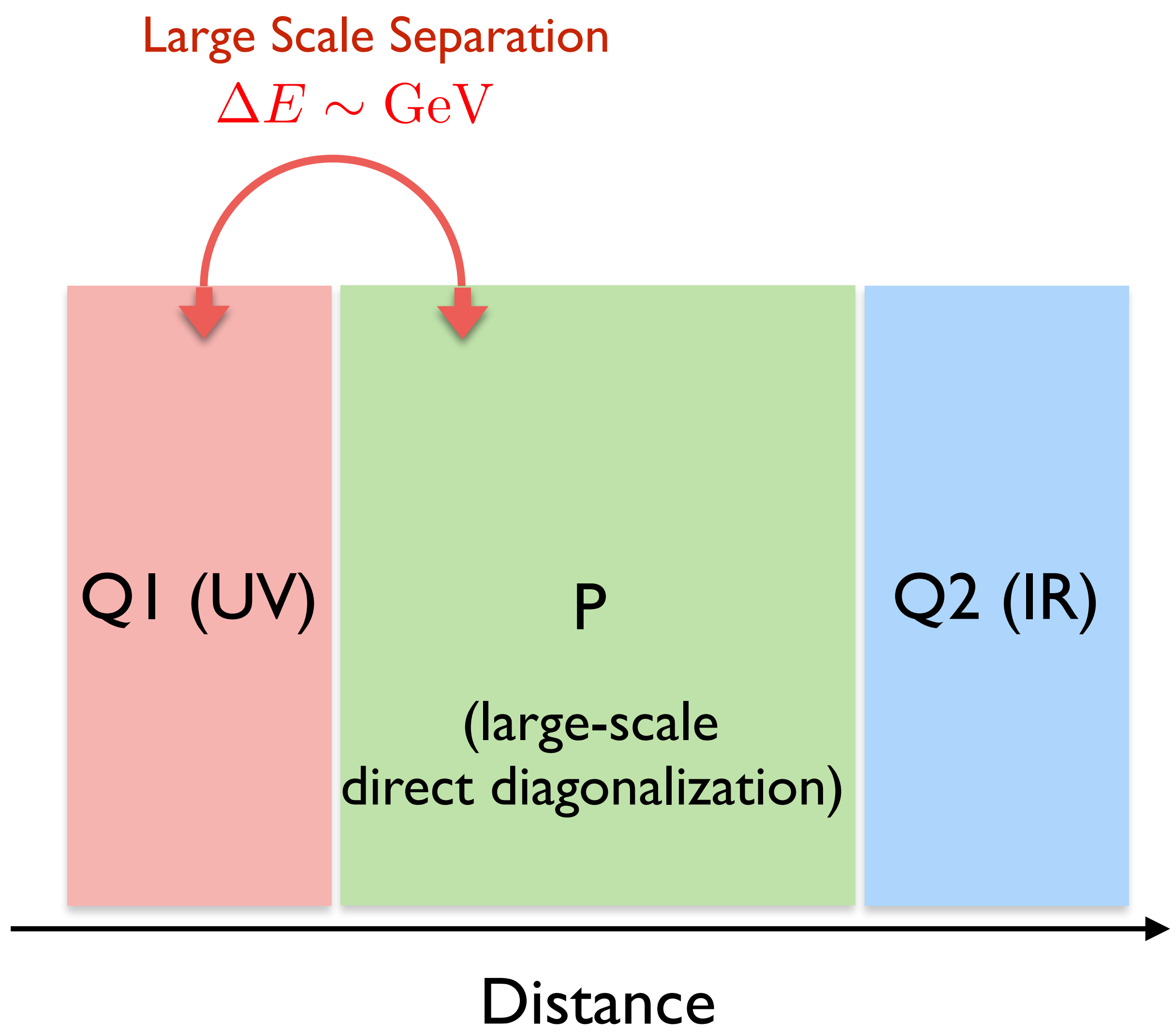
Coupling between P and Q2 is via the K.E. operator

$\vec{\nabla}^2$ connects neighboring shells

this means small energy denominators, highly energy dependent corrections

must be treated - but can be quasi-analytically

IR propagation enhanced because nuclei barely bound



Coupling between P and Q1 is via short-range strong interactions

Large energy denominators: energy independent corrections

Can be treated by a standard short range expansion

HOBET Formulation

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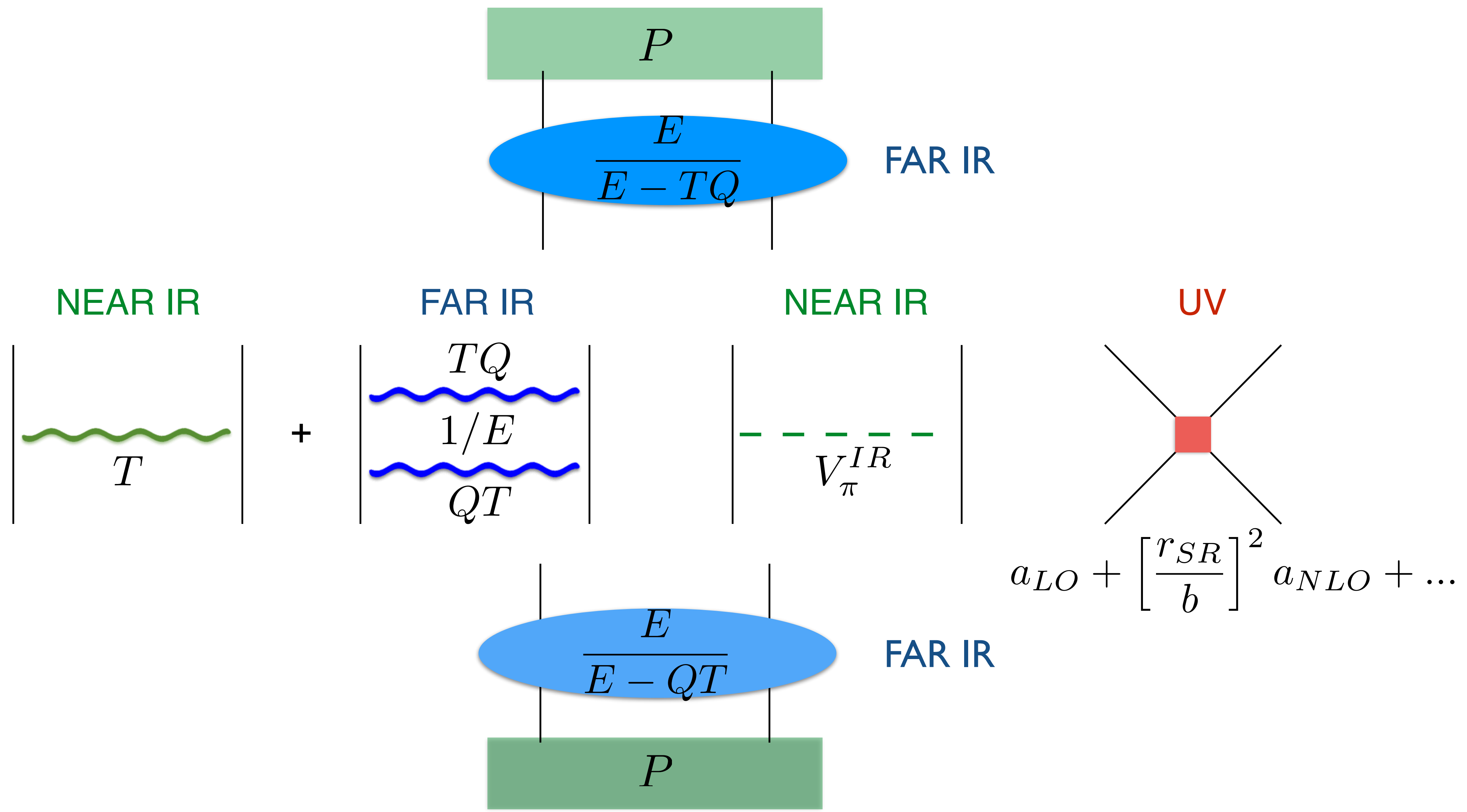
- Build the effective theory: $VG_{QH}QV \rightarrow V_\delta$ $\qquad V \rightarrow \begin{cases} 0 & \text{pionless} \\ V_\pi^{\rm IR} & \text{pionful} \end{cases}$

HOBET Interaction



Reviewing: Properties of HOBET

- One step from the UV theory of QCD $\rightarrow$ SM scale directly

- The choice of P is defined by parameters that the theorist is free to pick

$$P \rightarrow P(b, \Lambda)$$

  Rate of convergence may depend on the parameter choice, but not answers

- Gives the projection of the true solution to the HO for bound or continuum states

- Simple evolution: If one increments Λ , new configurations are added, but old ones remain the same

- Infinite number of solutions from a finite diagonalization: in particular, continuum solutions as a continuous function of E

- There is no reference to a potential in Q. The information previously encoded in and decoded from NN potentials now will be used directly in the SM-like space. How is that done?

Fixing the LECs: HOBET's short-range expansion is one in HO quanta:

$$(a_x^\dagger, a_y^\dagger, a_z^\dagger):\qquad a_i \equiv \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial r_i} + r_i\right) \qquad a_i \equiv \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial r_i} + r_i\right)$$

$$\boldsymbol{r} = \frac{1}{\sqrt{2}b}(\boldsymbol{r}_1 - \boldsymbol{r}_2) \qquad a_M^\dagger = \hat{e}_M \cdot \boldsymbol{a}^\dagger \qquad \tilde{a}_M = (-1)^M a_{-M}$$

- From these operators one can construct nodal and angular momentum raising and lowering operators

$$\tilde{\mathbf{a}} \odot \tilde{\mathbf{a}}\ |n\ell m\rangle = -2\sqrt{(n-1)\,(n+\ell-1/2)}\ |n-1\,\ell m\rangle$$

$$[[\tilde{\mathbf{a}} \otimes \tilde{\mathbf{a}} \otimes \cdots \otimes \tilde{\mathbf{a}}]_\ell \otimes |n\ell\rangle]_{00} = (-1)^\ell\, 2^{\ell/2} \sqrt{\frac{l!}{(2\ell-1)!!}\, \frac{\Gamma[n+\ell+\frac{1}{2}]}{\Gamma[n+\frac{1}{2}]}}\ |n00\rangle$$

- The expansion is effectively one around $r \sim b$

- Expansion order is defined in terms of oscillator quanta

$$V_\delta^S = a_{LO}^{\rm S}\delta(\boldsymbol{r}) + a_{NLO}^{\rm S}\left(\boldsymbol{a}^\dagger \odot \boldsymbol{a}^\dagger\ \delta(\boldsymbol{r}) + \delta(\boldsymbol{r})\ \tilde{\boldsymbol{a}} \odot \tilde{\boldsymbol{a}}\right) + \cdots$$

$$\delta(\boldsymbol{r}) \equiv \sum_{n'n} d_{n'n}^{00} |n'00\rangle\langle n00| \qquad\qquad d_{n'n}^{00} \equiv \frac{2}{\pi^2}\left[\frac{\Gamma(n'+\frac{1}{2})\Gamma(n+\frac{1}{2})}{(n'-1)!(n-1)!}\right]^{1/2}$$

$$\langle n'(\ell'=0S)JM;TM_T|V_\delta^{\rm S}|n(\ell=0S)JM;TM_T\rangle = d_{n'n}^{00}\left[a_{LO} - 2[(n'-1)+(n-1)]a_{NLO}^{\rm S} + \cdots\right]$$

- If we had computed the LECs from a potential, we would have found that the LECs are a non-local generalization of the familiar Talmi integrals

$$\int d\boldsymbol{r}'d\boldsymbol{r}\ r^{2p'}e^{-r'^2/2}Y_{00}(\Omega')\,V(\boldsymbol{r}',\boldsymbol{r})\,r^{2p}e^{-r^2/2}Y_{00}(\Omega)$$

$$a_{LO} \leftrightarrow (p',p) = (0,0) \qquad a_{NLO} \leftrightarrow (p',p) = (0,1)\ \text{or}\ (1,0) \qquad \text{etc.}$$

HOBET's V^{IR}_{π}

- V_π is already regulated by b, Λ
- our LO expansion systematically subtracts out the shortest range Talmi integrals: there is a 1-to-1 correspondence between LECs and Talmi integrals
- it makes no sense to include V_π in any Talmi integral that has an fitted LEC

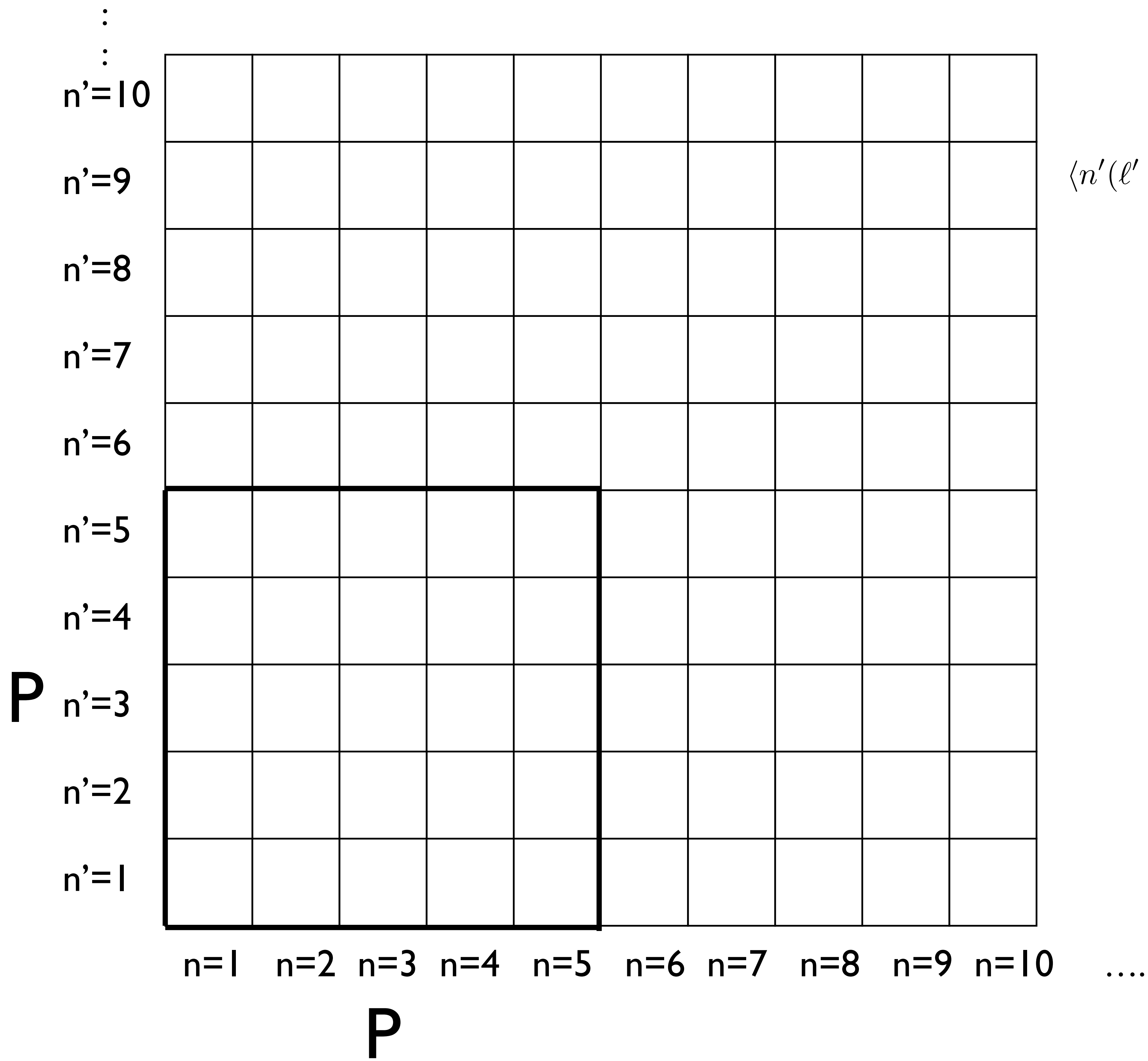
$$V_\pi = V_\pi^{IR} + V_\pi^{UV} \qquad\qquad V_\pi^{UV} = V_\delta(a_{LECs} \rightarrow a^\pi_{LECs})$$

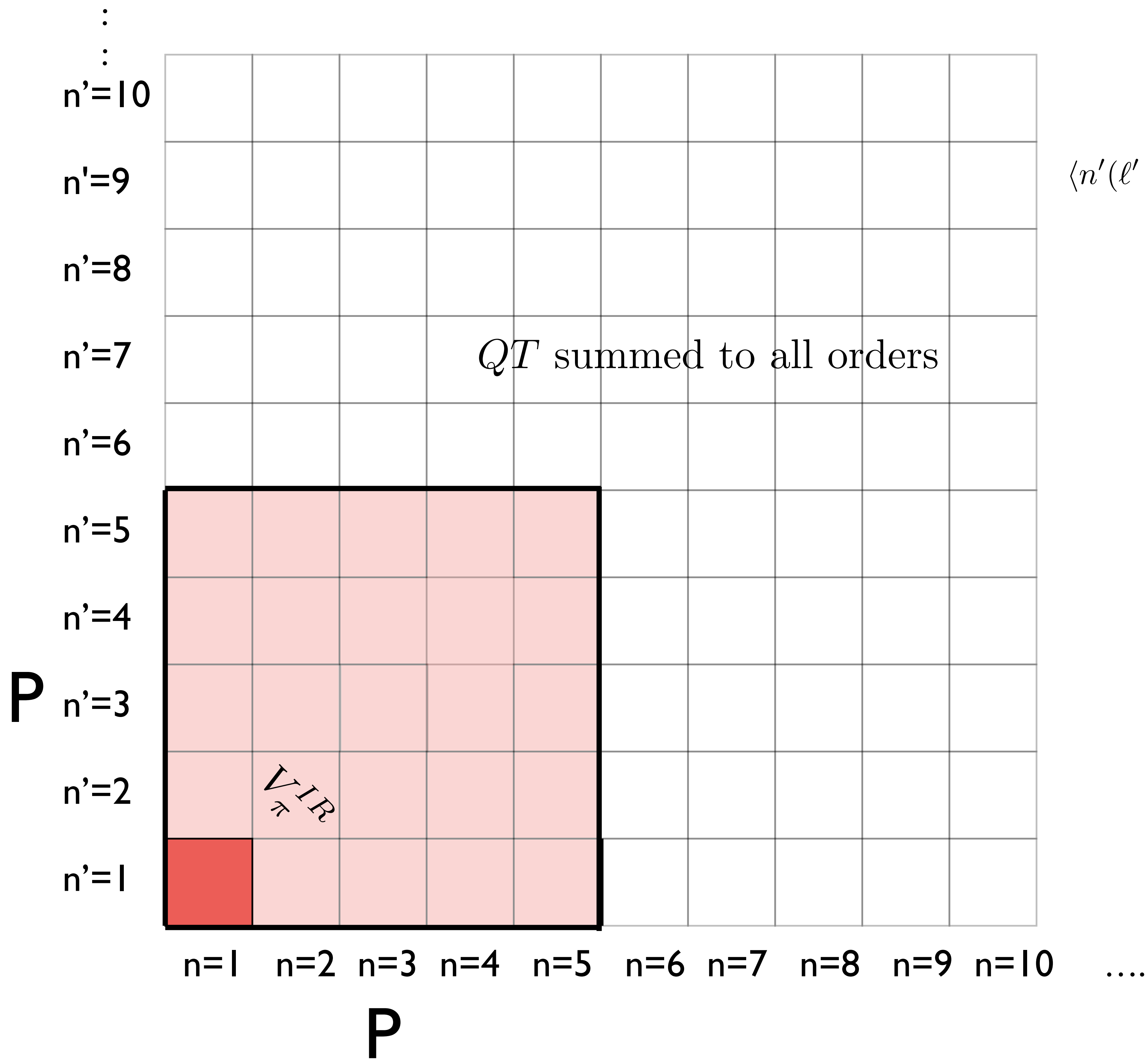
thus the only effect of V_π^{IR} is to correct the long-range Talmi integrals for which there is no LEC

- thus in HOBET the pion is a near-infrared contribution, weak and perturbative: its peak contribution (b=1.7 f) is at 4.1 f

$$\Lambda = 8\hbar\omega$$

$$\langle n'(\ell' = 0S)J;T|\ H^{\text{eff}}\ |n(\ell = 0S)J;T\rangle$$

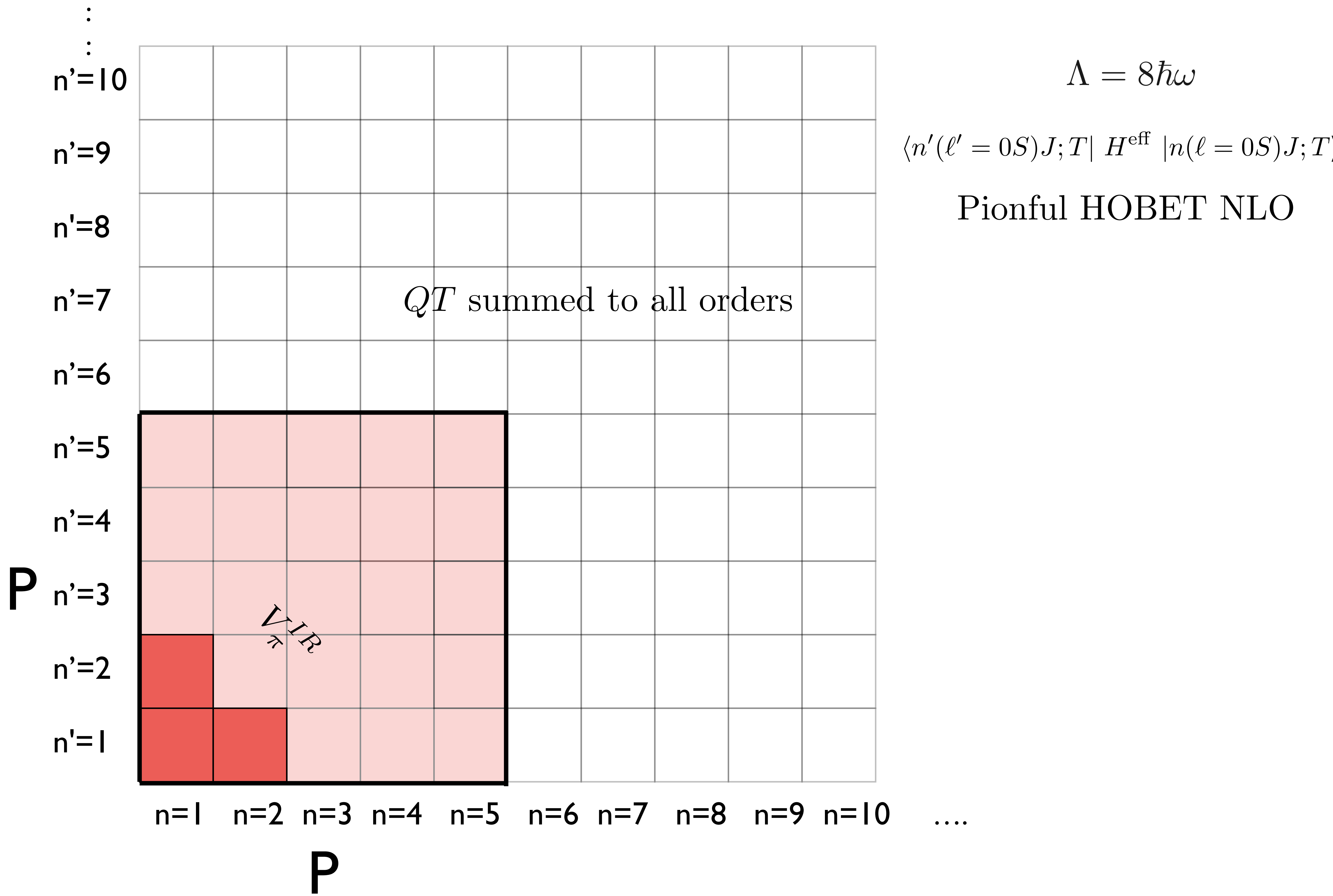




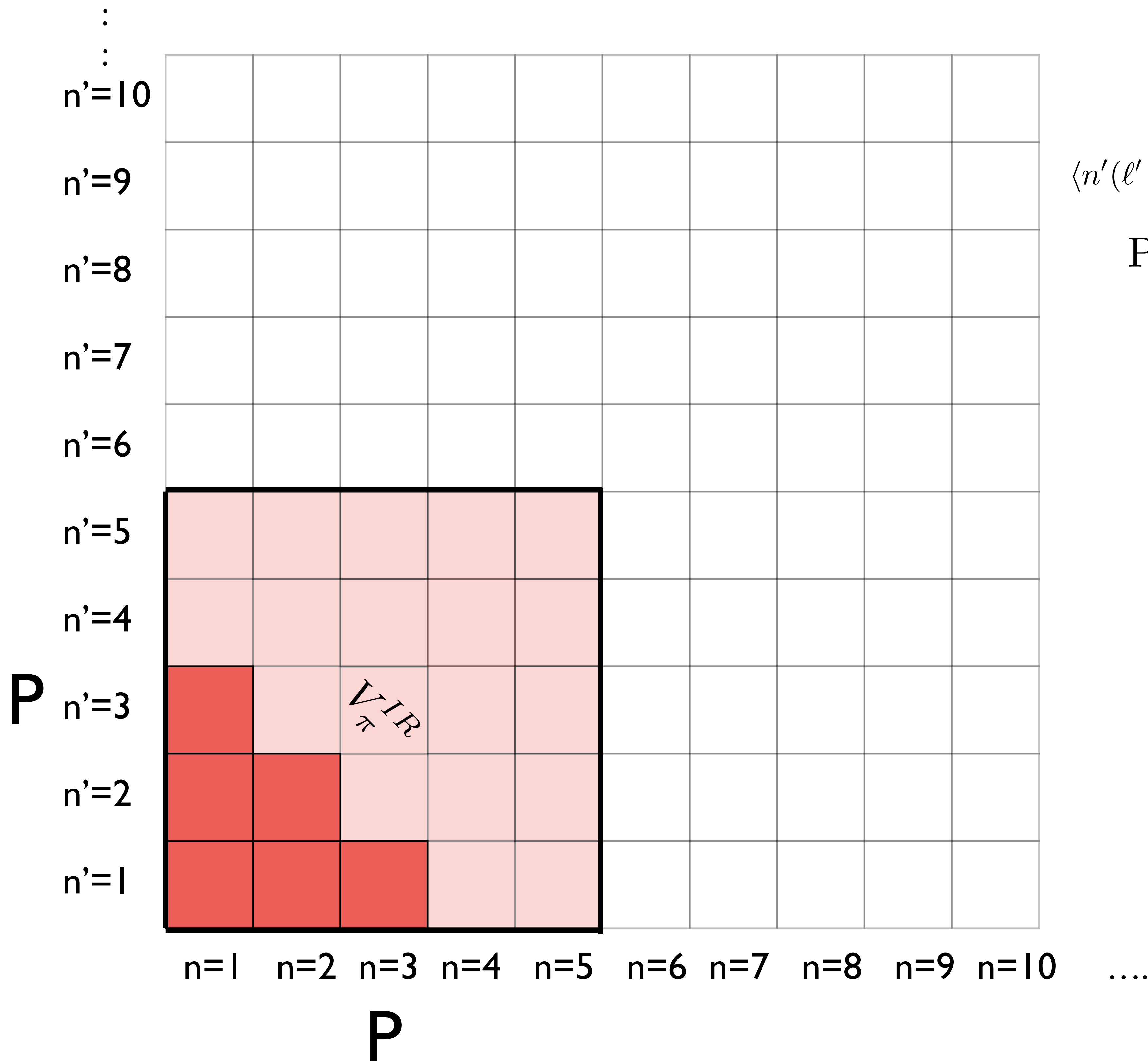
$\Lambda = 8\hbar\omega$

$\langle n'(\ell' = 0S)J;T|\ H^{\rm eff}\ |n(\ell = 0S)J;T\rangle$

Pionful HOBET LO



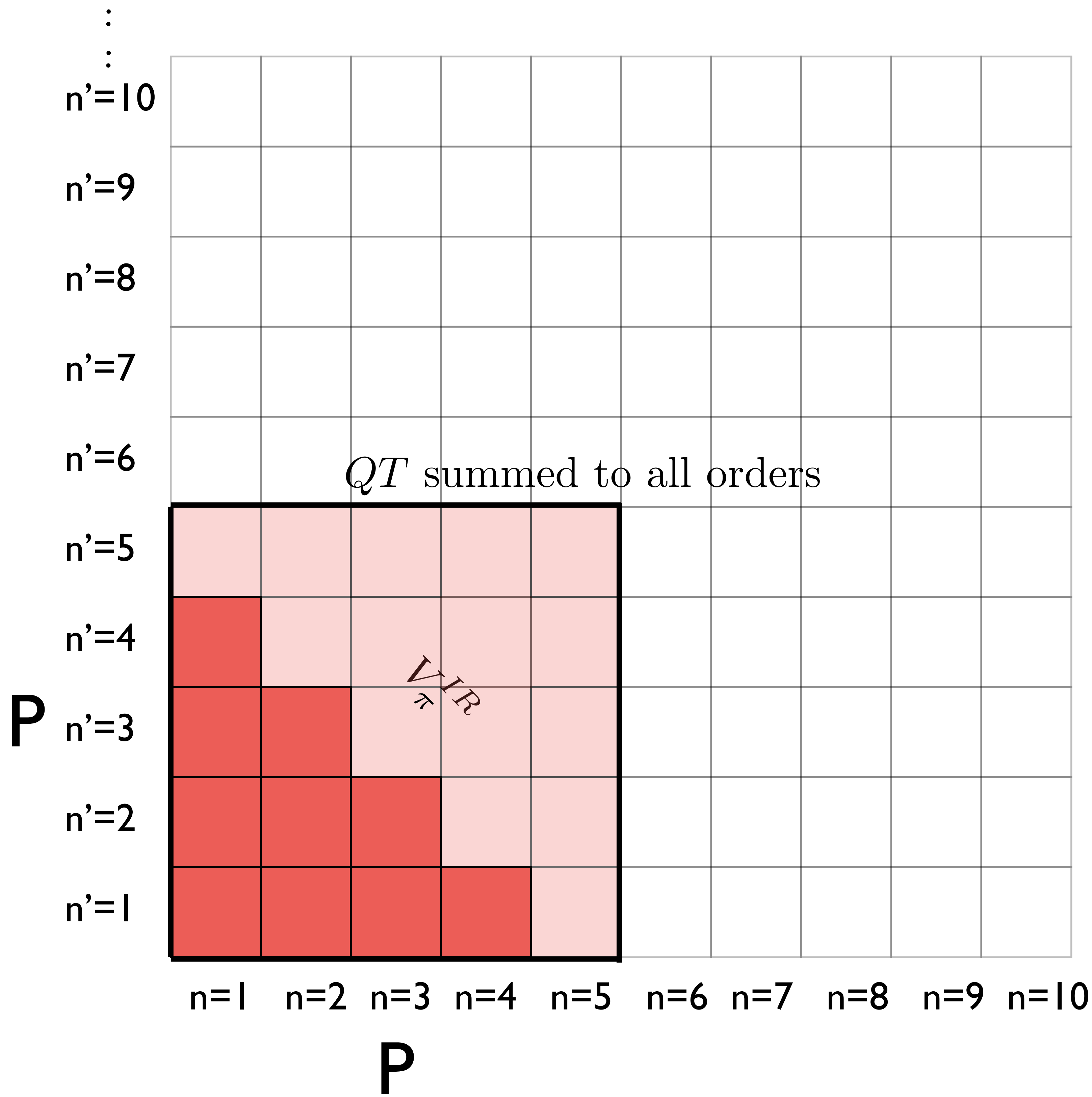
$\Lambda = 8\hbar\omega$
$\langle n'(\ell'=0S)J;T|\ H^{\mathrm{eff}}\ |n(\ell=0S)J;T\rangle$
Pionful HOBET NLO
QT summed to all orders
V_π^{IR}
P
P
n'=10
n'=9
n'=8
n'=7
n'=6
n'=5
n'=4
n'=3
n'=2
n'=1
n=1 n=2 n=3 n=4 n=5 n=6 n=7 n=8 n=9 n=10 ....



$\Lambda = 8\hbar\omega$

$\langle n'(\ell'=0S)J;T|\ H^{\rm eff}\ |n(\ell=0S)J;T\rangle$

Pionful HOBET N^2LO



$$\Lambda = 8\hbar\omega$$

$$\langle n'(\ell' = 0S)J;T|\ H^{\mathrm{eff}}\ |n(\ell = 0S)J;T\rangle$$

Pionful HOBET $\mathrm{N^3LO}$

As the short-range expansion is continued, pionic contributions are pushed further and further into the infrared

e.g., $^1S_0$

- Leading near-IR pionic contribution is governed by

$$\int d^3r\ r^8 e^{-r^2} \frac{e^{-\alpha r}}{\alpha r} \qquad r_{12} \equiv \sqrt{2} b r \qquad \alpha \equiv \frac{\sqrt{2} m_\pi c^2 b}{\hbar c} \sim 1$$

and the depends on a single dimensionless parameter

- The short-range operator structure at N^3LO is

$$\begin{aligned}
V_\delta^S = \sum_{n'n} \big[& a_{LO} |n'0\rangle\langle n0| \\
& + a_{NLO}^S [\boldsymbol{a}^\dagger \odot \boldsymbol{a}^\dagger |n'0\rangle\langle n0| + |n'0\rangle\langle n0| \tilde{\boldsymbol{a}} \odot \tilde{\boldsymbol{a}}] \\
& + a_{NNLO}^{S,22} \boldsymbol{a}^\dagger \odot \boldsymbol{a}^\dagger |n'0\rangle\langle n0| \tilde{\boldsymbol{a}} \odot \tilde{\boldsymbol{a}} \\
& + a_{NLO}^{S,40} [(\boldsymbol{a}^\dagger \odot \boldsymbol{a}^\dagger)^2 |n'0\rangle\langle n0| + |n'0\rangle\langle n0| (\tilde{\boldsymbol{a}} \odot \tilde{\boldsymbol{a}})^2] \\
& + a_{N^3LO}^{S,42} [(\boldsymbol{a}^\dagger \odot \boldsymbol{a}^\dagger)^2 |n'0\rangle\langle n0| \tilde{\boldsymbol{a}} \odot \tilde{\boldsymbol{a}} + \boldsymbol{a}^\dagger \odot \boldsymbol{a}^\dagger |n'0\rangle\langle n0| (\tilde{\boldsymbol{a}} \odot \tilde{\boldsymbol{a}})^2] \\
& + a_{N^3LO}^{S,60} [(\boldsymbol{a}^\dagger \odot \boldsymbol{a}^\dagger)^3 |n'0\rangle\langle n0| + |n'0\rangle\langle n0| (\tilde{\boldsymbol{a}} \odot \tilde{\boldsymbol{a}})^3] \big]
\end{aligned}$$

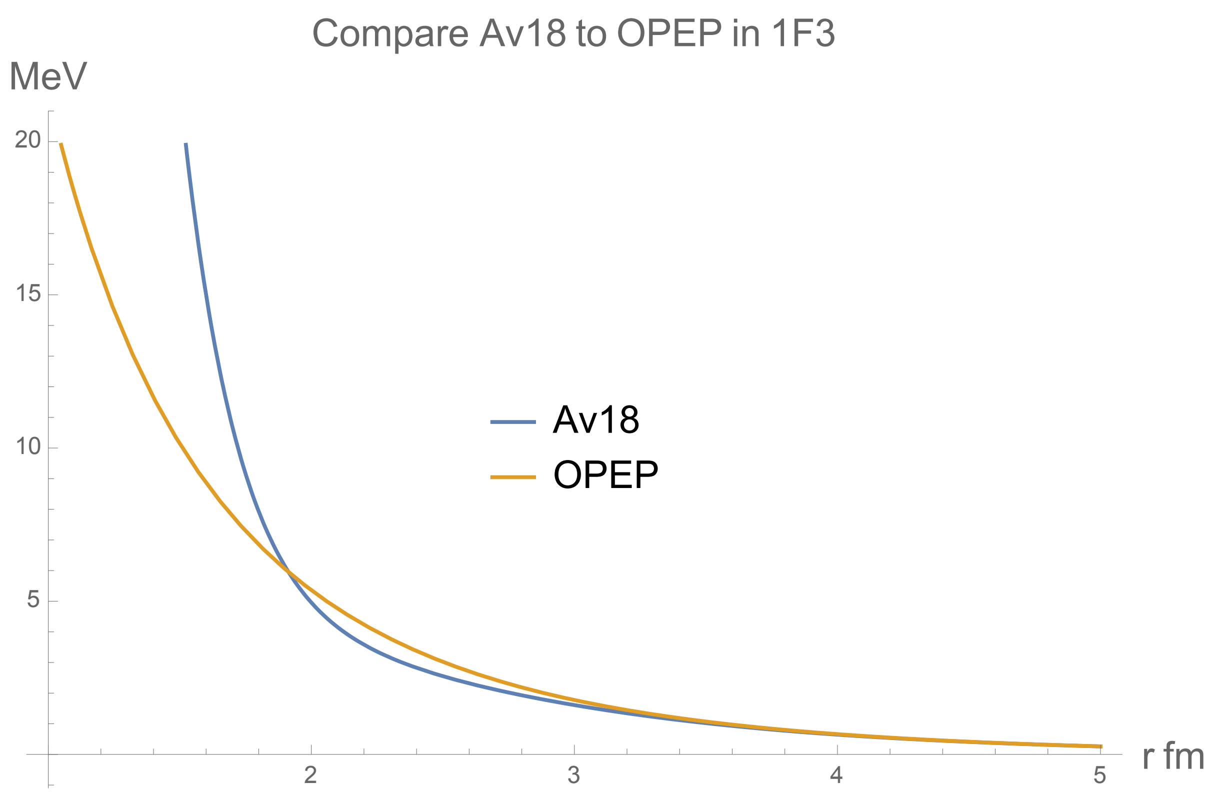
$$\langle n'(\ell'=0\ S)JM;TM_T|V_\delta^S|n(\ell=0\ S)JM;TM_T\rangle = d_{n'n}\big[a^S_{LO} - 2[(n'-1)+(n-1)]a^S_{NLO}$$
$$+4(n'-1)(n-1)a^{S,22}_{NNLO} + 4[(n'-1)(n'-2)+(n-1)(n-2)]a^{S,40}_{NNLO}$$
$$-8[(n'-1)(n'-2)(n-1)+(n'-1)(n-1)(n-2)]a^{S,42}_{N^3LO}$$
$$-8[(n'-1)(n'-2)(n'-3)+(n-1)(n-2)(n-3)]a^{60,S}_{N^3LO}]$$

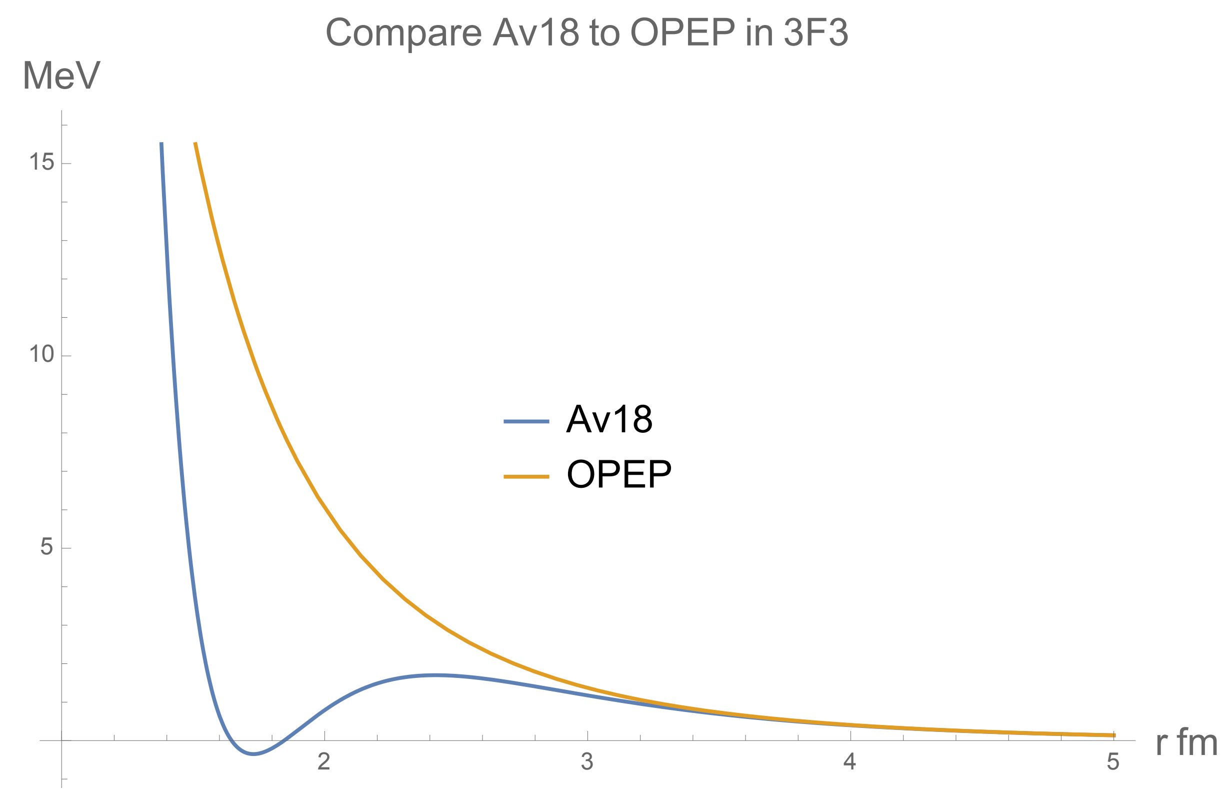
so $n'=1 \leftrightarrow n=1$ only gets a contribution from a_{LO}

and $n'=1 \leftrightarrow n=2$ gets contribution from a_{LO}, a_{NLO}

so scheme-independent fitting procedure

more generally, the lowest-energy information determines the LECs





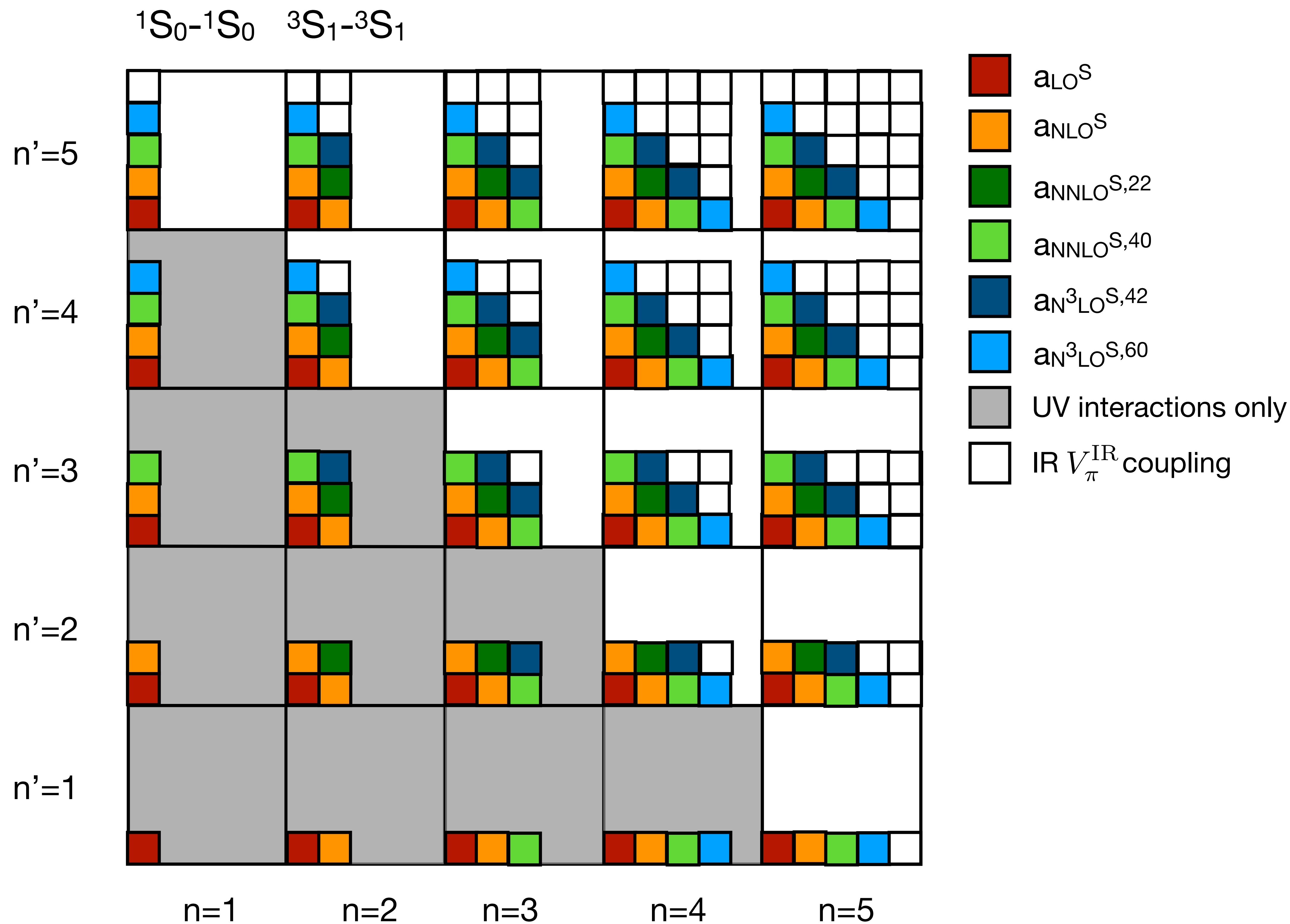
counting in singlet and triplet channels, pionful theory

implicit dimensionless parameter $\left(\frac{a_{SR}}{b}\right)^2$

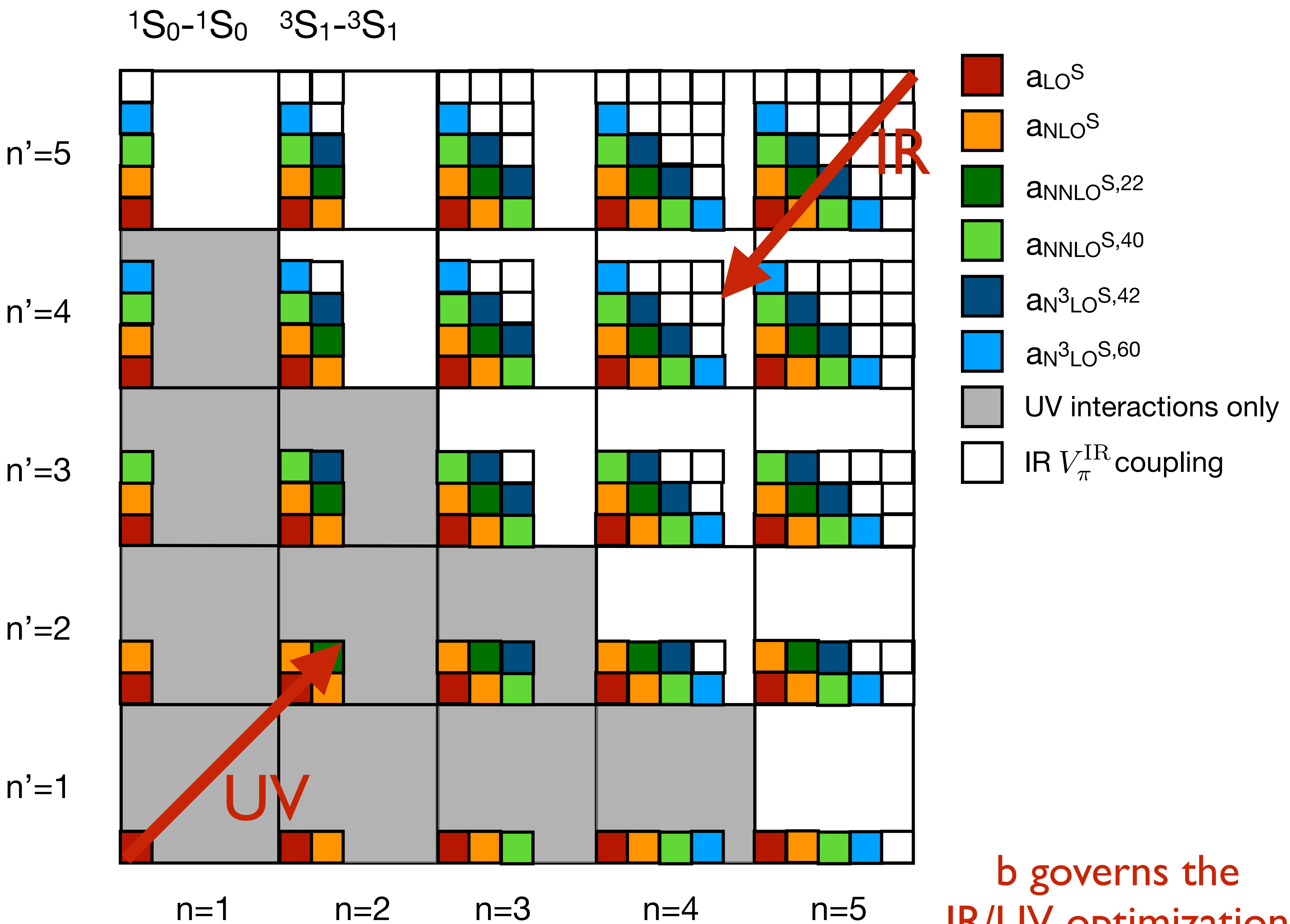
singlet $a_{SR} \sim 0.39f$

triplet $a_{SR} \sim 0.75f$

so typically an order of magnitude improvement per order

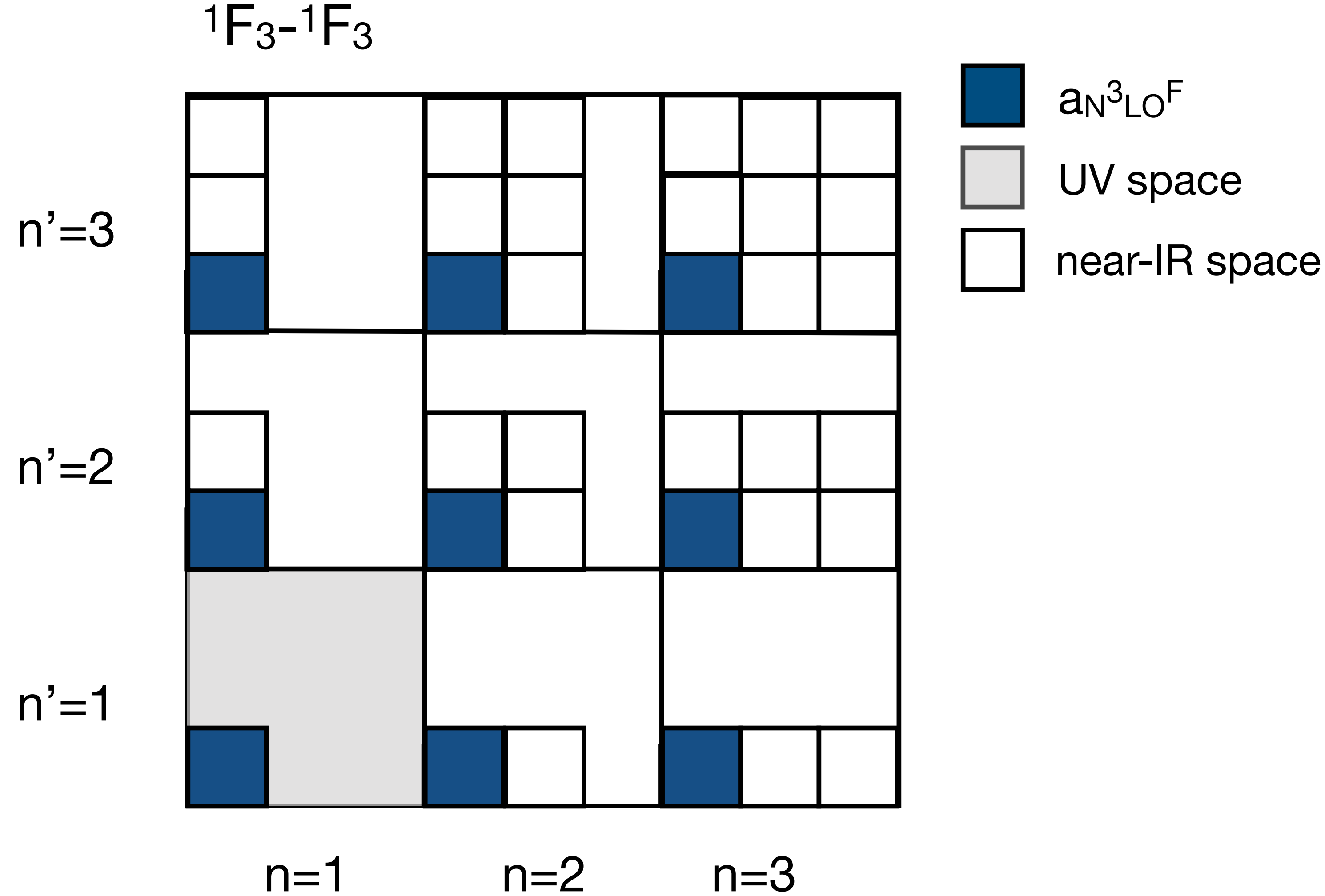


$^1S_0$-$^1S_0$
$^3S_1$-$^3S_1$
n'=5
n'=4
n'=3
n'=2
n'=1
n=1
n=2
n=3
n=4
n=5
$a_{LO}{}^S$
$a_{NLO}{}^S$
$a_{NNLO}{}^{S,22}$
$a_{NNLO}{}^{S,40}$
$a_{N^3LO}{}^{S,42}$
$a_{N^3LO}{}^{S,60}$
UV interactions only
IR $V_\pi^{\rm IR}$ coupling



b governs the IR/UV optimization

UV fitting



$$V_\pi = \frac{1}{12\pi}\left(\frac{g_A}{\sqrt{2}f_\pi}\right)^2 \tau_1\cdot\tau_2\left[m_\pi^3\frac{e^{-m_\pi r_{12}}}{m_\pi r_{12}}\sigma_1\cdot\sigma_2 + \text{ tensor } + \text{ contact }\right]$$

$$a_{N^3LO}^{\,^1F_3} \sim \left(\frac{g_A}{\sqrt{2}f_\pi}\right)^2 \int d^3\boldsymbol{r}\; r^6\;\frac{e^{-\alpha r}}{\alpha r} \qquad \alpha = \frac{\sqrt{2}m_\pi c^2 b}{\hbar c} \sim 1$$

$$\left(\frac{g_A}{\sqrt{2}f_\pi}\right)^2\Bigg|_{\text{fitted}} \sim (1\pm 0.02)\left(\frac{g_A}{\sqrt{2}f_\pi}\right)^2\Bigg|_{\text{cannonical}}$$

Reflects the simplicity of HOBET's IR Physics

Two pion exchange in the isoscalar $^1F_3$ channel, modeled with $m_\sigma \sim 615 \text{ MeV}$ for T=0

$$\frac{\langle V_{2\pi}^{IR} \rangle}{\langle V_{\pi}^{IR} \rangle} \sim 2.18 \left(\frac{m_N}{m_\pi} \right)^2 \frac{\int d^3r \; r^{(6-8)} \frac{e^{-\alpha_\sigma r}}{r}}{\int d^3r \; r^{(6-8)} \frac{e^{-\alpha_\pi r}}{r}} \qquad \alpha(m) \equiv \frac{\sqrt{2} mc^2 b}{\hbar c} = \begin{cases} 1.68 \\ 7.49 \end{cases}$$

yielding corrections to $a_{N^3LO}^{^1F_3}$ of 0.06% and to the first IR correction of 0.003%: despite the relativistic enhancement of the two-pion contribution, $V_{2\pi}$

These ratios can be evaluated analytically as a function of b, and remain valid unless b is taken to unnaturally small values - at N^{3}LO

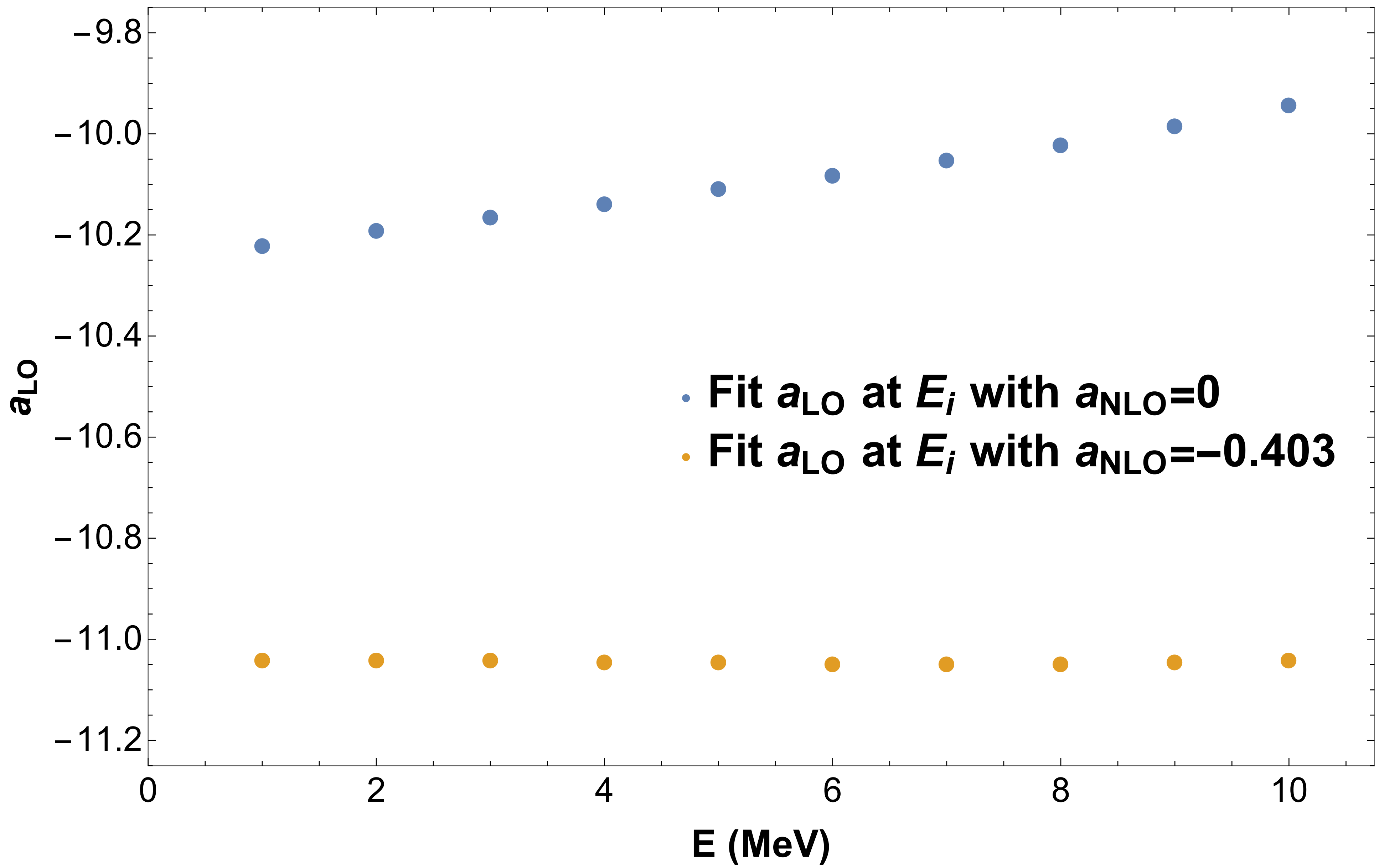
At LO, however, the correction is 25%: physically obvious as one is demanding that the IR correction treat everything beyond the expansion range. Very different from chiral potentials that have to treat the pion at all ranges, where the pionic physics becomes rapidly more complex with order

Similar conclusions follow for the polarization term $\langle V_\pi \frac{1}{E - QT} Q V_\pi \rangle$

Applying HOBET: the LEC fitting

- The BH equation must be solved self-consistently: the energy in = eigenvalue out
- Consider a bound-state: the Green's function $G_{QT}(E)$ depends only on E
- So iterate on E until self-consistency is achieved: typically 5 iterations
- You have the bound-state eigenvalue: if it is not the physical value, adjust aLO until the eigenvalue is correct — you have fixed an LEC
- But phase shifts: they carry most of the information. How do continuum states work?
- Pick an E: there will be a solution at that E. But $G_{QT}(E, \delta(E))$ - insert the experimental phase shift. Demand a self consistent solution at the chosen E: adjust the LECs systematically to achieve this

Ken McElvain developed an elegant way to do all of this that automatically identifies the relevant phase shift data — in typical SM spaces, up to about 120 E_{lab}



At any order, there is omitted physics from higher orders that becomes increasing important at E increases. This "contamination" is basically an uncertain encoded almost entirely in the last included order - illustrated here. This can be exploited to form a "cost function" that selects out the optimal range of phase shift data.

This establishes something quite important

- The HOBET interaction is quite energy depend due to the KE Green's functions $G_{QT}(E)$
- But all the LECs are constant
- $VG_{QH}QV$ does depend on E, but weakly due to the large scale separation: this energy dependence can be absorbed through the momentum dependence of the operators
- The only residual strong-interaction energy dependence is the one generic to all EFTs — the neglect of the next order
- This all the wonderful properties of an energy-dependent EFT interaction are realized, yet the interaction is as simple as the standard energy-independent ones

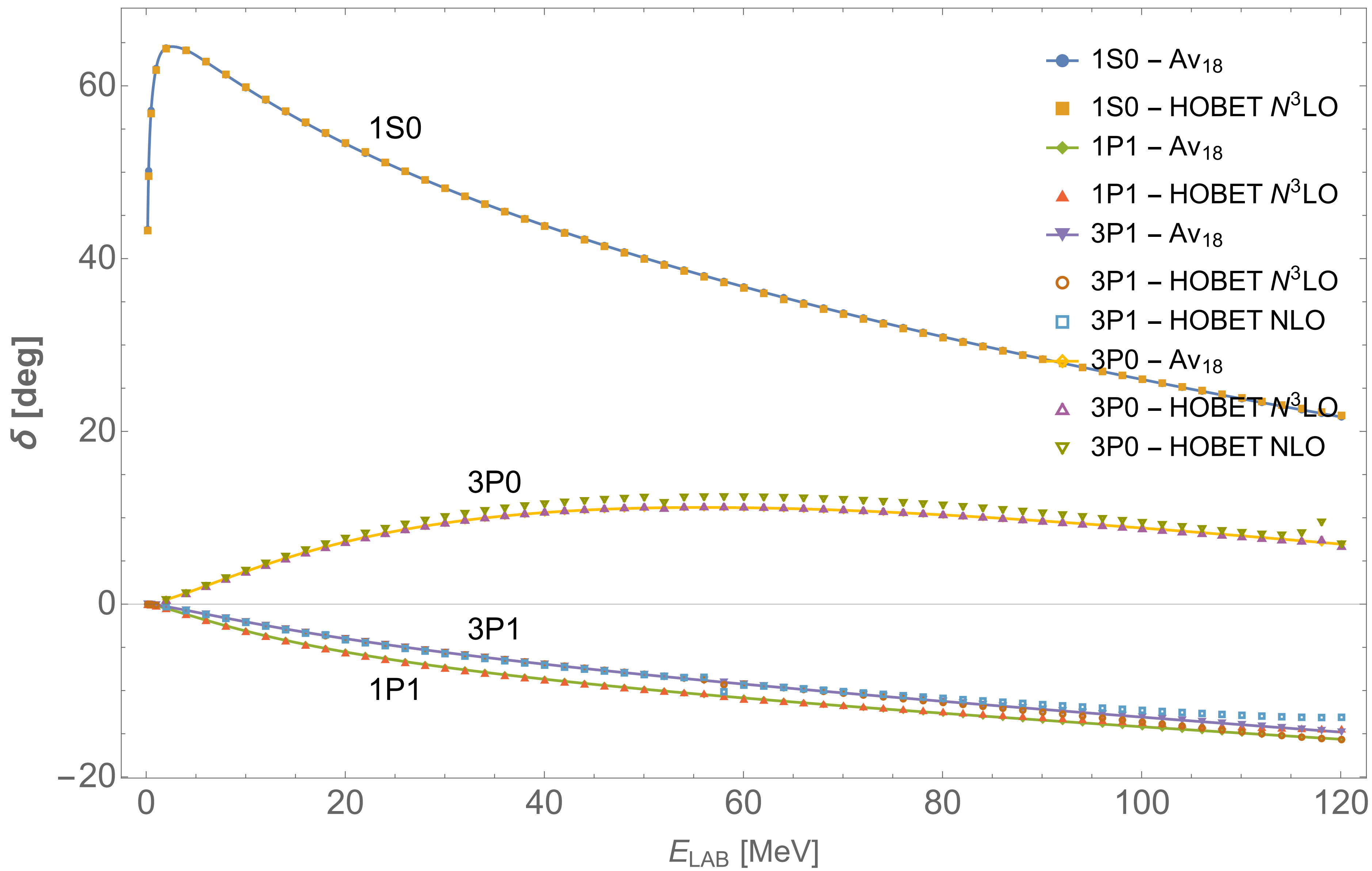
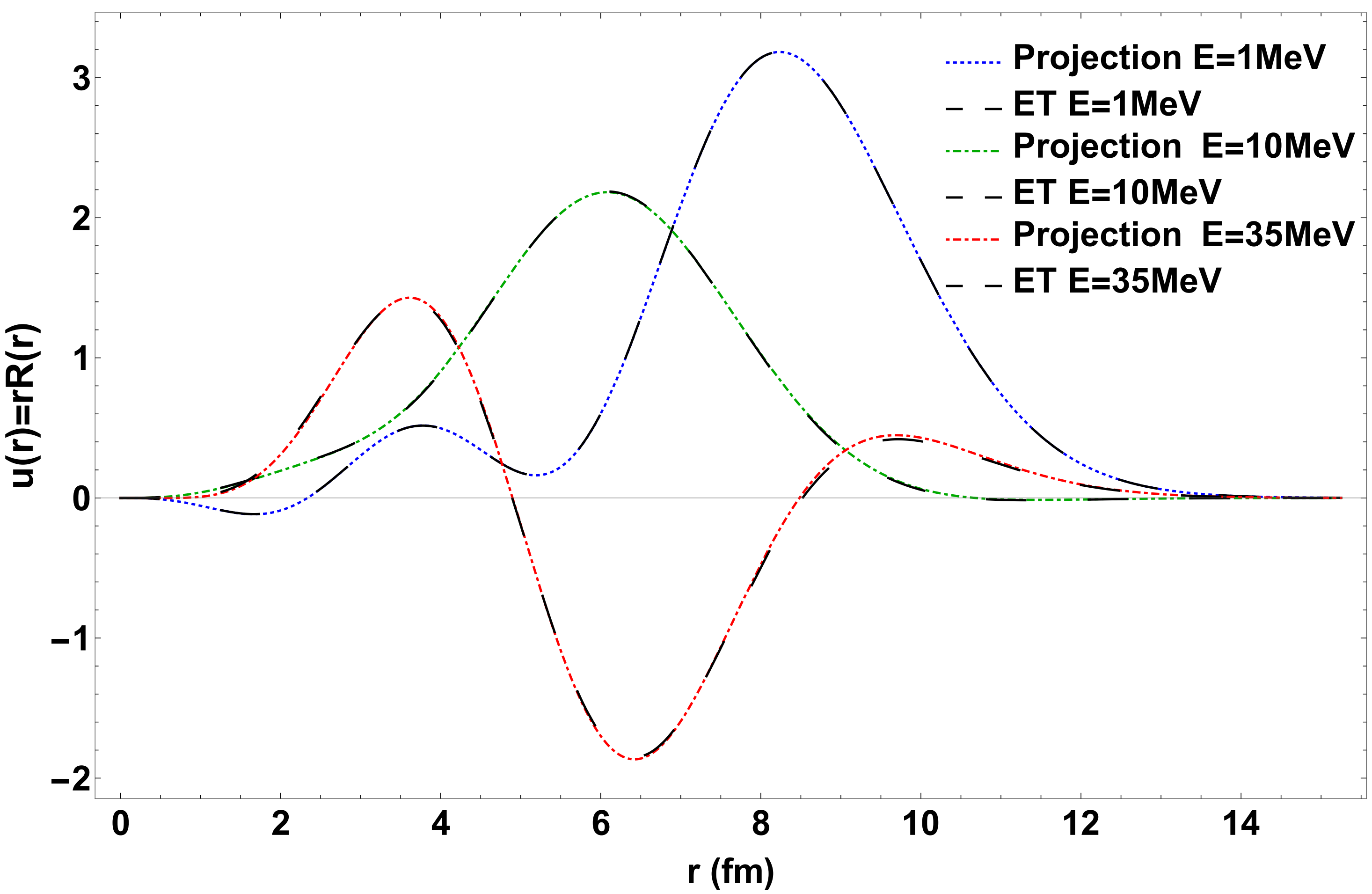


Fig. 4. Phase shifts regenerated from LECs fit to data from 1 to 80 MeV and compared to the original phase shifts from Av$_{18}$. In the $^1S_0$ channel the low energy behavior down to 50 keV associated with a resonance at ~ 74 keV is reproduced from data above 1 MeV. In the $^3P_0$ and $^3P_1$ channels even NLO results based on a single LEC reproduce phase shifts quite well.

TABLE I. Deuteron channel: binding energy E_b as a function of the expansion order. Bare denotes a calculation with $T+V$
.

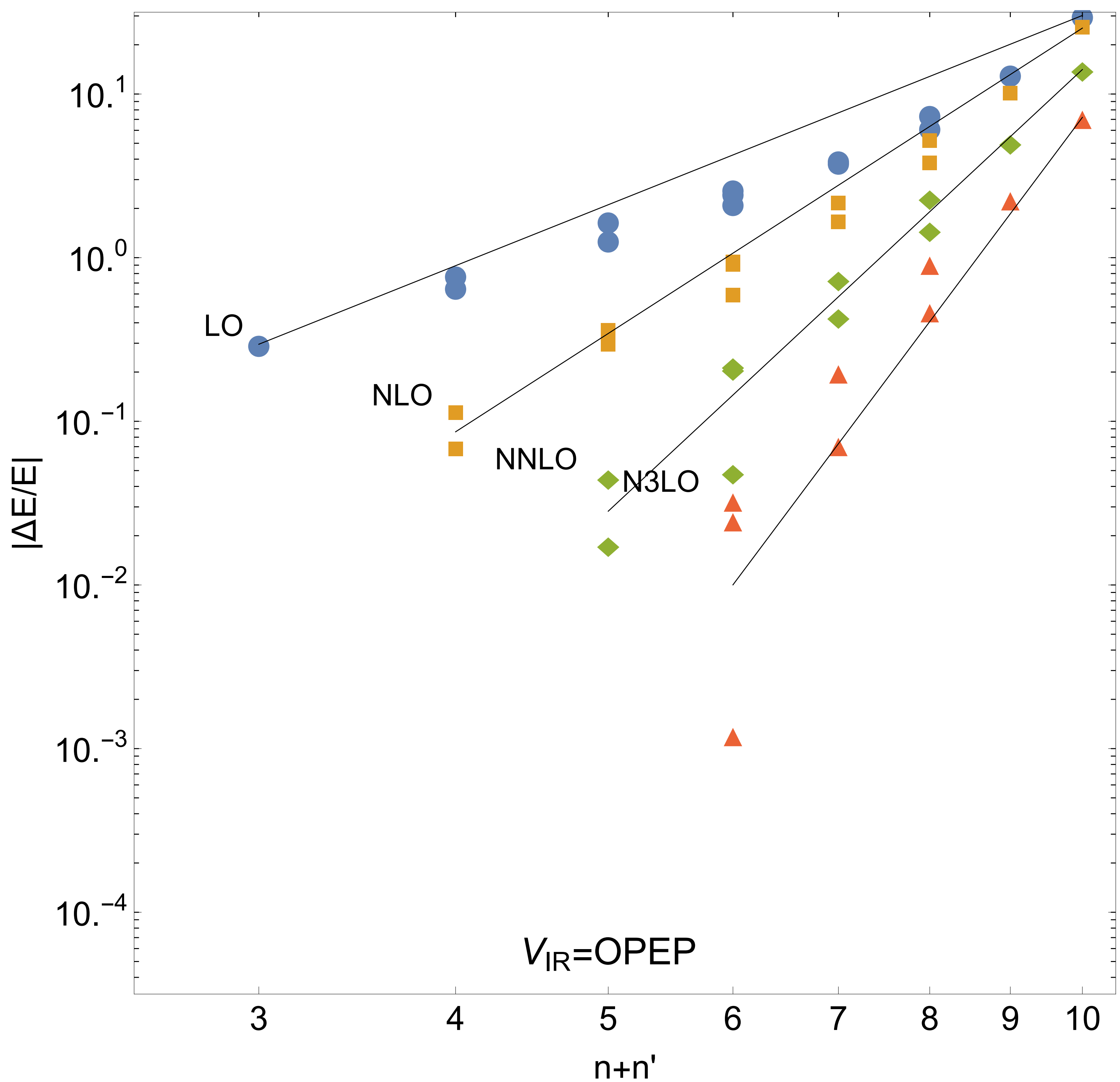
Order	$E_{\mathrm{b}}^{\mathrm{pionless}}$	$C^2\ (LECs)$	$E_{\mathrm{b}}^{\mathrm{pionful}}$	$C^2\ (LECs)$
bare	3.09525	-	-0.76775	-
LO	-1.27715	2.2E-2	-2.01110	1.9E-3
NLO	-1.95424	1.6E-2	-2.19833	2.2E-6
NNLO	-2.17307	6.7E-3	-2.21705	4.0E-8
$\mathrm{N^3LO}$	-2.23175	1.3E-3	-2.22464	8.4E-9

Virtual perfect to the scattering data for E_{CM} 0-40 MeV: pionful HOBET accurate to 0.1 keV

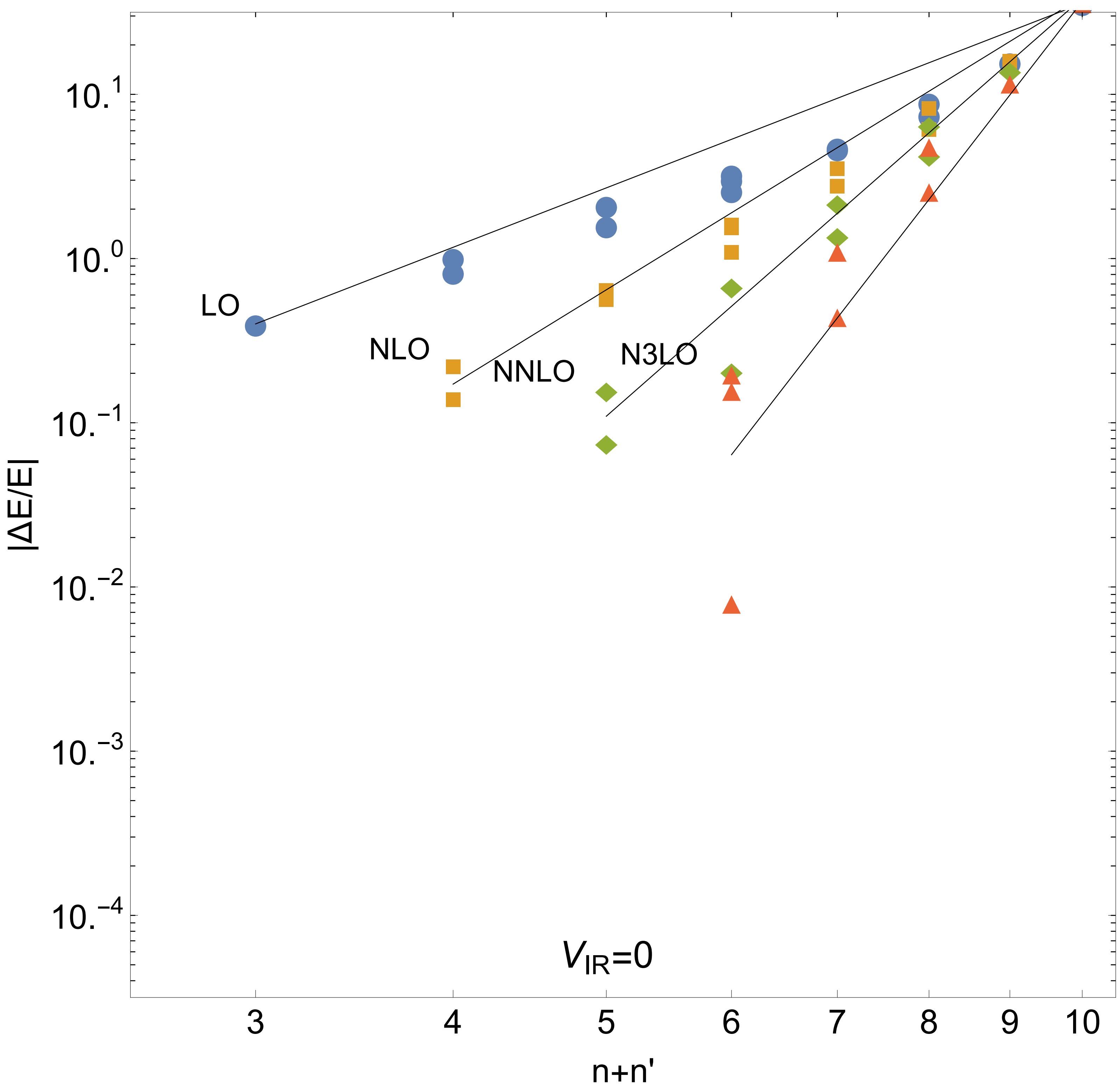


ıPı $P|\Psi\rangle$

Continuous function of E, r reproduced virtually exactly with 4 LECs



Lepage Plot for Scheme-Independent Fitting: Pionful, Phase Shifts Only



Lepage Plot for Scheme-Independent Fitting: Pionless, Phase Shifts Only

Summary and Next Steps

* HOBET is highly predictive, converges at nuclear momentum scales, and rapidly improves with order — just as one wants from an effective theory

* Pionful theory better than pionless, but both converge well

* There is a systematic "counting scheme" behind the expansion, connected with the treatment of the pion: it has the nice property that as the order of the UV expansion increases, the role of (say) two-pion corrections decreases, relatively to the simple tree-level contribution

* Three-body terms are now being evaluated

* These interactions are those needed in N-body work: convergence depends on the expectation that the number of nucleons interacting through strong interactions at one time declines rapidly with that number